MEASURING THE REDISTRIBUTIVE IMPACT OF TAXES AND TRANSFERS IN THE PRESENCE OF RERANKING

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ABSTRACT

This paper provides a theoretical foundation for analyzing the redistributive effect of taxes and transfers for the case in which the ranking of individuals by pre-fiscal income changes as a result of fiscal redistribution. Through various examples, this paper shows how reranking--a common feature in all actual fiscal systems--reduces the predictive power of simple measures of progressivity in assessing the actual effect of taxes and transfers on inequality.

JEL Codes: H22, D31, A23.
Keywords: Marginal contribution, vertical equity, reranking.

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1. Introduction

In Chapter 2 by Enami, Lustig, and Aranda, we discussed how to measure the redistributive impact of taxes and transfers in a system where there is no reranking: i.e., the position of individuals ordered by their income remains identical in the pre-fiscal and post-fiscal situations.[1] This paper introduces the possibility of reranking in a fiscal system into the analysis of a tax or transfer's marginal contribution in reducing (increasing) inequality. As will become clear in this paper, when a fiscal system creates reranking in individuals, it is much harder to use simple rules to determine whether a specific tax or transfer is equalizing or not. The complicated math introduced here shows that, in contrast to such measures as progressivity, the marginal contribution analysis is the only straightforward way of determining whether a tax or transfer is equalizing. It should be noted that the analysis in this paper is focused on the traditional Gini index but can be similarly extended to the S-Gini indexes. The idea of marginal contribution analysis can be also extended to other measures of inequality but one should be cautious about the fact that the type of decomposition that we use in this paper may not be applicable to other indexes (for example, the Theil index).

The best way to see how introducing reranking would create new problems is through a simple example. In chapter 2 of the CEQ Handbook[2] reranking was not present, a simple rule was introduced that held that if a system has only one tax and that tax is progressive, then the post-fiscal system is unambiguously more equal. Though this "progressive-means-equalizing" rule of thumb is one of the most commonly used rules, the aforementioned chapter showed that this rule is not always correct when a system is not composed of only one tax or one transfer (see for example, the so-called Lambert conundrum). This paper shows that in the presence of reranking, this rule is not always correct even in a system with only one tax (transfer). In other words, this paper shows that a progressive tax could create a more unequal post-fiscal system (using Gini as the measure of inequality). Table 1 shows an example where the Gini increases from 0.054 to 0.074 after introducing a progressive tax into the system.

Table 1. Example of an Unequalizing Progressive Tax in the Presence of Reranking

Individual	**Original Income**	**Tax**	**End Income**
1	10.00	0.00	10.00
2	11.00	2.00	9.00
3	12.00	4.00	8.00
4	13.00	6.00	7.00
Total	**46.00**	**12.00**	**34.00**
Average	**11.50**	**3.00**	**8.50**
Gini	**0.0540**	**n.c.**	**0.0740**

n.c. Not calculated for the purposes of this paper.

[1] Enami and others (2017).
[2] Enami and others (2017).

Before continuing further, the following section will explain the notations that will be used throughout this paper.

2. Notations

This section provides the definitions of notations that will be used throughout this paper. The notations are generally similar to those in other papers but some minor modifications have been made to meet the requirements of the topics covered here.

2.1. Gini and Concentration Coefficients

This paper uses G_Q and C_Q^G for the "Gini coefficient of the income concept Q" and the "concentration coefficient of income concept Q with respect to the income concept G." Note how the Gini and concentration coefficients are calculated using the covariance formula:

$$G_Q = \frac{2cov(Q, F_Q)}{\mu_Q}$$

and

$$C_Q^G = \frac{2cov(Q, F_G)}{\mu_Q}$$

where, F_Q is the normalized rank of individuals when they are ranked by income concept *Q* and μ_Q is the average value of the income concept *Q*. The normalized rank is simply calculated as follows. Assume there are *n* individuals who are ranked by income *Q* from 1 to *n*, where *n* is the rank of the individual with the highest income. The normalized rank of individual *j* is simply equal to j / n. Therefore, the normalized rank ranges from $1 / n$ to 1. Similarly, F_G is the normalized rank of individuals if they are ranked by income concept *G*.

The aforementioned chapter uses a simpler notation, C_Q, for the concentration coefficient, which implies that the "original income ranking of households" is used in its calculation. This paper uses the superscript *X* to represent that individuals are ranked by their original income:

$$C_Q = C_Q^X = \frac{2cov(Q,F_X)}{\mu_Q}.$$

The covariance formula helps to explain why the concentration coefficient can be negative. For example, if the ranking of individuals is exactly the opposite with income concept *Q* than with income concept *X*, then C_Q^X would be negative. On the other hand, the Gini coefficient for income concept *Q*, G_Q is always non-negative since it uses the same income concept to calculate the Gini index as it uses to rank individuals.

2.2. Reynolds-Smolensky (R-S) and Kakwani Indexes

Similarly to section 2.1, I use the following formulas for the R-S and Kakwani indexes of a tax (T) or transfer (B) when they are calculated with respect to the original income ranking of households.

For a tax,

$$\Pi_T^{RS} = G_X - C_{X-T}^X = \frac{2cov(X, F_X)}{\mu_X} - \frac{2cov(X-T, F_X)}{\mu_X(1-g)}$$

$$\Pi_T^K = C_T^X - G_X = \frac{2cov(T, F_X)}{\mu_X g} - \frac{2cov(X, F_X)}{\mu_X}$$

For a transfer,

$$\rho_B^{RS} = G_X - C_{X+B}^X = \frac{2cov(X, F_X)}{\mu_X} - \frac{2cov(X+B, F_X)}{\mu_X(1+b)}$$

$$\rho_B^K = G_X - C_B^X = \frac{2cov(X, F_X)}{\mu_X} - \frac{2cov(B, F_X)}{\mu_X b}$$

where g (b) is the total taxes (transfers) collected divided by the total amount of original income (that is, *X*). For example,

$$g = \frac{T}{X}$$

and

$b = \frac{B}{X}$.

In this paper, I also use a modified version of these two indicators (the R-S and Kakwani indexes) that allows the basis for ranking to be different from the original income. Whenever I use these new indexes, the superscript shows the income concept for the ranking. For example, if I used income concept *Q* for the ranking, I would have the following formulas.

For a tax,

$$\Pi_T^{RS^Q} = C_X^Q - C_{X-T}^Q$$

$$\Pi_T^{K^Q} = C_T^Q - C_X^Q$$

For a transfer,

$$\rho_B^{RS^Q} = C_X^Q - C_{X+B}^Q$$

$$\rho_B^{K^Q} = C_X^Q - C_B^Q$$

The relationship between the R-S and Kakwani indexes is as follows.

For a tax,

$$\Pi_T^{RS^Q} = \frac{g}{1-g}\Pi_T^{K^Q}.$$

And for a transfer,

$$\rho_B^{RS^Q} = \frac{b}{1+b}\rho_B^{K^Q}.$$

2.3. The Relationship Between the Redistributive Effect, Vertical Equity, and Reranking

To understand how reranking affects a fiscal system, it is helpful to decompose the redistributive effect (RE), which is the change in Gini from the original income to the end income, into the vertical equity (VE) and the reranking (RR) components. The following derivation shows explicitly that RR always reduces VE and is therefore always an unequalizing component. The presence of RR in a fiscal system implies a form of inefficiency in redistributive policy because the same level of reduction in inequality could be achieved with a lower level of income redistribution through taxes and transfers if RR were to be eliminated.

For the purpose of simplicity, I bundle all of the taxes in a system together and all of transfers (benefits) together and use just one tax (T) and one transfer (B) in the following.

The RE (that is, the change in Gini) can be decomposed into two elements,[3] as follows:

[3] See Duclos and Araar (2007). Note that the component called VE in equation 2.3.1 is not exactly pure and could include a "horizontal inequality" component. This component captures the "negative" behavior of a fiscal system that treats differently individuals who are exactly the same (Duclos and Araar, 2007). Here it is assumed that people are not exactly the same, so the horizontal inequality does not exist. Note that the phrase "exactly the same" is not limited to the amount of original income and includes other elements such as number of children and even subjective measures. If people have exactly the same original income, the derivations here are still valid, so we assume people are not exactly the same in other dimensions but we allow them to have identical original income.

(1) $$G_X - G_{X-T+B} = \underbrace{(G_X - C^X_{X-T+B})}_{Vertical\ Equity} + \underbrace{(C^X_{X-T+B} - G_{X-T+B})}_{Reranking\ (non-positive)}\ .$$

These indexes are known as the Reynolds-Smolensky index of progressivity and VE[4] and the Atkinson-Plotnick index of RR.[5] According to Lambert,[6] in the absence of RR, the change in Gini can be simply calculated using the following formula (assuming only one tax and one transfer or, alternatively, grouping all taxes together as well as all transfers).

$$G_X - C^X_{X-T+B} = \frac{(1-g)\Pi^{RS}_T + (1+b)\rho^{RS}_B}{1-g+b}$$

If reranking is allowed, the change in Gini will be equal to

(2) $$\boldsymbol{G_X - G_{X-T+B} =}$$

$$\boldsymbol{\frac{(1-g)\Pi^{RS}_T+(1+b)\rho^{RS}_B}{1-g+b} + \left(G_X - C_X^{X-T+B}\right) - \left(\frac{(1-g)\left(\Pi^{RS}_T - \Pi^{RS}_T{}^{X-T+B}\right)+(1+b)\left(\rho^{RS}_B - \rho^{RS}_B{}^{X-T+B}\right)}{1-g+b}\right)}.$$

The proof is as follows.

We know that the change in Gini can be decomposed into two elements:

(3) $$G_X - G_{X-T+B} = (G_X - C^X_{X-T+B}) + (C_{X-T+B} - G^X_{X-T+B})$$

As mentioned previously, Lambert proves the following inequality:[7]

(4) $$G_X - C^X_{X-T+B} = \frac{(1-g)\Pi^{RS}_T+(1+b)\rho^{RS}_B}{1-g+b}\ .$$

Now, focusing on the second term in equation 3, that is, the RR term, we know from equation 4 that

(5)

$$C^X_{X-T+B} = G_X - \frac{(1-g)\Pi^{RS}_T+(1+b)\rho^{RS}_B}{1-g+b}.$$

Now, focusing on G_{X-T+B},

[4] Reynolds and Smolensky (2013).
[5] Atkinson (1979); Plotnick (1981, 1982).
[6] See Lambert (2001, p. 277).
[7] See Lambert (2001, p. 277).

(6) $$G_{X-T+B} = \frac{2Cov(X-T+B, F_{X-T+B})}{\mu_X(1-g+b)} \rightarrow$$

$$G_{X-T+B} = \frac{2Cov(X-T, F_{X-T+B})}{\mu_X(1-g+b)} + \frac{2Cov(X+B, F_{X-T+B})}{\mu_X(1-g+b)} - \frac{2Cov(X, F_{X-T+B})}{\mu_X(1-g+b)}$$

$$= \underbrace{\left(\frac{(1-g)}{(1-g+b)}\right)\frac{2Cov(X-T, F_{X-T+B})}{\mu_X(1-g)}}_{A} + \underbrace{\left(\frac{(1+b)}{(1-g+b)}\right)\frac{2Cov(X+B, F_{X-T+B})}{\mu_X(1+b)}}_{B}$$

$$\underbrace{-\frac{2Cov(X, F_{X-T+B})}{\mu_X(1-g+b)}}_{C}.$$

To make it simpler to follow the next steps, I examine each one of the three terms in equation 6 in turn.

$$A = \left(\frac{(1-g)}{(1-g+b)}\right)\frac{2Cov(X-T, F_{X-T+B})}{\mu_X(1-g)} - \left(\frac{(1-g)}{(1-g+b)}\right)\frac{2Cov(X, F_{X-T+B})}{\mu_X}$$

$$+\left(\frac{(1-g)}{(1-g+b)}\right)\frac{2Cov(X, F_{X-T+B})}{\mu_X}$$

Note that I just added and subtracted the same term in the preceding equation at the end. It is important to note that the first two terms in the preceding formula would add up to

$-\left(\frac{(1-g)}{(1-g+b)}\right)\Pi_T^{RS^{X-T+B}}$ (see the notation section).

The third term is equal to

$\left(\frac{(1-g)}{(1-g+b)}\right)C_X^{X-T+B}$.

Therefore,

(7) $$A = -\left(\frac{(1-g)}{(1-g+b)}\right)\Pi_T^{RS^{X-T+B}} + \left(\frac{(1-g)}{(1-g+b)}\right)C_X^{X-T+B}.$$

Analogously for B,

(8)

$$B = -\left(\frac{(1+b)}{(1-g+b)}\right)\rho_B^{RS^{X-T+B}} + \left(\frac{(1+b)}{(1-g+b)}\right)C_X^{X-T+B}.$$

And similarly for C,

(9)

$C = -\left(\frac{1}{(1-g+b)}\right) C_X^{X-T+B}.$

The following formula puts the preceding parts together (that is, it uses 7, 8, and 9 in equation 6).

(10)

$$G_{X-T+B} = A + B + C = -\left[\frac{(1-g)\Pi_T^{RS^{X-T+B}} - (1+b)\rho_B^{RS^{X-T+B}}}{1-g+b}\right] + C_X^{X-T+B}\ .$$

Finally, the following formula puts all the parts together (that is, it uses 4, 5, and 10 in 3).

$$\boldsymbol{G_X - G_{X-T+B} =}$$

$$\boldsymbol{\frac{(1-g)\Pi_T^{RS} + (1+b)\rho_B^{RS}}{1-g+b} + \left(G_X - C_X^{X-T+B}\right) - \left(\frac{(1-g)\left(\Pi_T^{RS} - \Pi_T^{RS^{X-T+B}}\right) + (1+b)\left(\rho_B^{RS} - \rho_B^{RS^{X-T+B}}\right)}{1-g+b}\right)}$$

Q.E.D.

It should be noted that since the RR term is always non-positive, the following expression is always negative:

$$\left(G_X - C_X^{X-T+B}\right) - \left(\frac{(1-g)\left(\Pi_T^{RS} - \Pi_T^{RS^{X-T+B}}\right) + (1+b)\left(\rho_B^{RS} - \rho_B^{RS^{X-T+B}}\right)}{1-g+b}\right) \leq 0$$

Also, equation 2 can be further simplified:

(11)

$$G_X - G_{X-T+B} = \left(G_X - C_X^{X-T+B}\right) + \left(\frac{(1-g)\Pi_T^{RS^{X-T+B}} + (1+b)\rho_B^{RS^{X-T+B}}}{1-g+b}\right)$$

2.4. Marginal Contribution

Based on equation 11, I can now derive the formula for the marginal contribution of a tax (or transfer).

For simplicity, I define income concepts Z and $Z\backslash T_1$ as follows:

$$Z = X - \sum_{i=1}^{n} T_i + \sum_{j=1}^{m} B_j$$
$$Z\backslash T_1 = X - \sum_{i=2}^{n} T_i + \sum_{j=1}^{m} B_j$$

In the general case, I define the marginal contribution of Tax 1 (without the loss of generality) as follows:

$$M_{T_1} = G_{Z\backslash T_1} - G_Z.$$

The interpretation of this formula is straightforward: the marginal contribution of a tax is equal to the change in the Gini index when this tax is added to the rest of the taxes and transfers in the system.

By adding and subtracting G_X in the above equation, we would have

$$M_{T_1} = G_{Z\backslash T_1} - G_Z + G_X - G_X$$

which can then be rewritten as

(12)

$$M_{T_1} = (G_X - G_Z) - \left(G_X - G_{Z\backslash T_1}\right).$$

Using a generalized version of equation 11, we can rewrite equation 12 as follows:

(13)

$$\boldsymbol{M_{T_1}} = \left\{\left(\boldsymbol{G_X} - \boldsymbol{C_X^Z}\right) + \left(\frac{\sum_{i=1}^{n}(1-g_i)\Pi_{T_i}^{RS^Z} + \sum_{j=1}^{m}(1+b_j)\rho_{B_j}^{RS^Z}}{1-\sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j}\right)\right\} - \left\{\left(\boldsymbol{G_X} - \boldsymbol{C_X^{Z\backslash T_1}}\right) + \left(\frac{\sum_{i=2}^{n}(1-g_i)\Pi_{T_i}^{RS^{Z\backslash T_1}} + \sum_{j=1}^{m}(1+b_j)\rho_{B_j}^{RS^{Z\backslash T_1}}}{1-\sum_{i=2}^{n} g_i + \sum_{j=1}^{m} b_j}\right)\right\}.$$

Similarly, the marginal contribution of a benefit can be defined as follows:

(14)

$$\boldsymbol{M_{B_1}} = \left\{\left(\boldsymbol{G_X} - \boldsymbol{C_X^Z}\right) + \left(\frac{\sum_{i=1}^{n}(1-g_i)\Pi_{T_i}^{RS^Z} + \sum_{j=1}^{m}(1+b_j)\rho_{B_j}^{RS^Z}}{1-\sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j}\right)\right\} - \left\{\left(\boldsymbol{G_X} - \boldsymbol{C_X^{Z\backslash B_1}}\right) + \left(\frac{\sum_{i=1}^{n}(1-g_i)\Pi_{T_i}^{RS^{Z\backslash B_1}} + \sum_{j=2}^{m}(1+b_j)\rho_{B_j}^{RS^{Z\backslash B_1}}}{1-\sum_{i=1}^{n} g_i + \sum_{j=2}^{m} b_j}\right)\right\}.$$

Note that derivations 13 and 14 use a modified R-S index that ranks individuals by income concepts other than by the original income. One can suggest alternative formulas that are based on the ranking with respect to the original income. The following examples provide such derivations.

Beginning with equation 12,

$$M_{T_1} = (G_X - G_Z) - \left(G_X - G_{Z\backslash T_1}\right)$$

$$= [(G_X - C_Z^X) + (C_Z^X - G_Z)] - \left[\left(G_X - C_{Z\backslash T_1}^X\right) + \left(C_{Z\backslash T_1}^X - G_{Z\backslash T_1}\right)\right]$$

we can rearrange the above terms to have

$$M_{T_1} = \overbrace{\left[(G_X - C_Z^X) - \left(G_X - C_{Z\backslash T_1}^X\right)\right]}^{Contribution\ of\ T_1\ to\ vertical\ equity} + \overbrace{\left[(C_Z^X - G_Z) - \left(C_{Z\backslash T_1}^X - G_{Z\backslash T_1}\right)\right]}^{Contribution\ of\ T_1 to\ reranking}.$$

Using the relationship between VE and the R-S index of the taxes and transfers (calculated with respect to the original income ranking of households), we can rewrite the above equation as follows:

$$M_{T_1} = \left[\left(\frac{\sum_{i=1}^{n}(1-g_i)\Pi_{T_i}^{RS} + \sum_{j=1}^{m}(1+b_j)\rho_{B_j}^{RS}}{1-\sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j}\right) - \left(\frac{\sum_{i=2}^{n}(1-g_i)\Pi_{T_i}^{RS} + \sum_{j=1}^{m}(1+b_j)\rho_{B_j}^{RS}}{1-\sum_{i=2}^{n} g_i + \sum_{j=1}^{m} b_j}\right)\right]$$

$$+ \underbrace{\left[(C_Z^X - G_Z) - \left(C_{Z\backslash T_1}^X - G_{Z\backslash T_1}\right)\right]}_{Contribution\ of\ T_1 to\ reranking}.$$

Now, simplifying the above equation we have

(15)

$$\boldsymbol{M_{T_1}} = \left[\left(\frac{\left[\left(\mathbf{1}-\sum_{i=2}^{n}\boldsymbol{g_i}+\sum_{j=1}^{m}\boldsymbol{b_j}\right)(\mathbf{1}-\boldsymbol{g_1})\Pi_{T_1}^{RS}\right]+\left[(\boldsymbol{g_1})\left(\sum_{i=2}^{n}(\mathbf{1}-\boldsymbol{g_i})\Pi_{T_i}^{RS}+\sum_{j=1}^{m}(\mathbf{1}+\boldsymbol{b_j})\boldsymbol{\rho}_{B_j}^{RS}\right)\right]}{\left(\mathbf{1}-\sum_{i=1}^{n}\boldsymbol{g_i}+\sum_{j=1}^{m}\boldsymbol{b_j}\right)\left(\mathbf{1}-\sum_{i=2}^{n}\boldsymbol{g_i}+\sum_{j=1}^{m}\boldsymbol{b_j}\right)}\right)\right] + \underbrace{\left[(C_Z^X-G_Z)-\left(C_{Z\setminus T_1}^X-G_{Z\setminus T_1}\right)\right]}_{\textit{Contribution of } T_1 \textit{ to reranking}},$$

which can be also written as follows:

(16)

$$\boldsymbol{M_{T_1}} = \left[\left(\frac{\left[(\mathbf{1}-\boldsymbol{g_1})\Pi_{T_1}^{RS}\right]+\left[(\boldsymbol{g_1})\overbrace{\left(\boldsymbol{G_X}-\boldsymbol{C_{Z\setminus T_1}^X}\right)}^{\textit{VE of the system without } T_1}\right]}{\left(\mathbf{1}-\sum_{i=1}^{n}\boldsymbol{g_i}+\sum_{j=1}^{m}\boldsymbol{b_j}\right)}\right)\right] + \underbrace{\left[(C_Z^X-G_Z)-\left(C_{Z\setminus T_1}^X-G_{Z\setminus T_1}\right)\right]}_{\textit{Contribution of } T_1 \textit{ to reranking}}.$$

Similarly, for a transfer we have the following formulas:

(17)

$$\boldsymbol{M_{B_1}} = \left[\left(\frac{\left[\left(\mathbf{1}-\sum_{i=1}^{n}\boldsymbol{g_i}+\sum_{j=2}^{m}\boldsymbol{b_j}\right)(\mathbf{1}+\boldsymbol{b_1})\boldsymbol{\rho}_{B_1}^{RS}\right]-\left[(\boldsymbol{b_1})\left(\sum_{i=1}^{n}(\mathbf{1}-\boldsymbol{g_i})\Pi_{T_i}^{RS}+\sum_{j=2}^{m}(\mathbf{1}+\boldsymbol{b_j})\boldsymbol{\rho}_{B_j}^{RS}\right)\right]}{\left(\mathbf{1}-\sum_{i=1}^{n}\boldsymbol{g_i}+\sum_{j=1}^{m}\boldsymbol{b_j}\right)\left(\mathbf{1}-\sum_{i=1}^{n}\boldsymbol{g_i}+\sum_{j=2}^{m}\boldsymbol{b_j}\right)}\right)\right] + \underbrace{\left[\left(\boldsymbol{C_Z^X}-\boldsymbol{G_Z}\right)-\left(\boldsymbol{C_{Z\setminus B_1}^X}-\boldsymbol{G_{Z\setminus B_1}}\right)\right]}_{\textit{Contribution of } B_1 \textit{ to reranking}}$$

Or

(18)

$$\boldsymbol{M_{B_1}} = \left[\left(\frac{\left[(\boldsymbol{1}+\boldsymbol{b_1})\boldsymbol{\rho_{B_1}^{RS}}\right] - \left[(\boldsymbol{b_1}) \overbrace{\left(\boldsymbol{G_X} - \boldsymbol{C_{Z\backslash B_1}^X}\right)}^{\boldsymbol{VE\ of\ the\ system\ without\ B_1}}\right]}{\left(\boldsymbol{1} - \sum_{i=1}^{n} \boldsymbol{g_i} + \sum_{j=1}^{m} \boldsymbol{b_j}\right)}\right)\right] + \underbrace{\left[\left(\boldsymbol{C_Z^X} - \boldsymbol{G_Z}\right) - \left(\boldsymbol{C_{Z\backslash B_1}^X} - \boldsymbol{G_{Z\backslash B_1}}\right)\right]}_{\boldsymbol{Contribution\ of\ B_1 to\ reranking}}.$$

In the rest of this paper, I rely mainly on equations 13, 15, and 16 for the analysis related to the marginal contribution of a tax, and 14, 17, and 18 for the analysis related to the marginal contribution of a transfer. Equations 13 and 14 give us a rule of thumb for cases of multiple taxes and transfers and for cases when the tax or transfer of interest does not change the end income ranking of individuals (as will become clearer later in this paper). These two equations, however, rely on the calculation of the R-S and Kakwani indexes with respect to the end income ranking of individuals, which is an inferior method to calculating them by the original income ranking because the indicators based on the end income ranking are dependent whereas the indicators based on the original income ranking are independent. In other words, any change in a tax (size, progressivity, introducing or removing a tax) can change the R-S index of a transfer if the end income ranking is used in the calculation of this index. Moreover, the aforementioned chapter[8] uses only the original income ranking, so using equations 15, 16, 17, and 18 would provide comparable results to chapters 1 and 2 in the CEQ Handbook. When there is no RR (as in chapter 2 from the CEQ Handbook), the value of the R-S and Kakwani indexes is the same no matter which ranking is used.

2.5. Vertical Equity

As in the previous papers, VE is defined as follows:

$$VE_Z = G_X - C_Z^X.$$

This formula uses the original income both as the starting point and as a basis for ranking, but we can generalize it to use any other income concept for the purpose of ranking:

$$VE_{L,M}^Q = C_L^Q - C_M^Q.$$

3. In the Presence of Reranking, Is the Marginal Contribution of a Tax Equalizing?

This section examines the marginal contribution of a tax and identifies conditions that make a tax equalizing. The conditions are derived for different scenarios, beginning with a system that only has one tax, then a system that has a transfer, and finally a system with multiple taxes and transfers (besides the specific tax that is of the interest of the analysis).

[8] Enami and others (2017).

3.1. The Case of Only One Tax

Although a progressive tax in a system with no-reranking is always equalizing, this is not the case when there is RR (see table 3 at the beginning of this paper). Since there is only one tax, equation 13 can be simplified as follows:

(19) $$M_T = (G_X - C_X^{X-T}) + \Pi_T^{RS^{X-T}}.$$

Using equation 16, we have the following:

(20) $$M_T = \Pi_T^{RS} + (C_{X-T}^{X} - G_{X-T}).$$

Because equation 20 is easier to use, I will focus on it. Note that the RR term is always non-positive, that is
$C_{X-T}^{X} - G_{X-T} \leq 0$.

For a tax to be equalizing, equation 20 has to be positive:

$$M_T = \Pi_T^{RS} + (C_{X-T}^{X} - G_{X-T}) > 0$$

or

(21)
$\Pi_T^{RS} > (G_{X-T} - C_{X-T}^{X})$
or

(22) $$\Pi_T^{K} > \left(\frac{1-g}{g}\right)(G_{X-T} - C_{X-T}^{X}).$$

Note that the right-hand side of equation 22 is always non-negative[9] and reaches its minimum (that is, zero) when the ranking of individuals before and after adding the tax remains the same. Therefore, a progressive tax (which is defined as a tax where $\Pi_T^K > 0$) is only equalizing when equation 22 holds. However, a regressive tax ($\Pi_T^K < 0$) is always unequalizing. Surprisingly, however, a neutral tax ($\Pi_T^K = 0$) can be unequalizing when it creates RR.

Table 2 identifies the effect of adding a tax to a system that has no other tax or transfer in place.

[9] This can be shown intuitively. For any income value, the deviation of highest and lowest income from the average and their rank from the average rank is the highest. The underlying covariance formula multiplies these deviations for each person and adds them together. Since Gini multiplies the largest deviation of income by the largest deviation of rank (for example, for a person with the highest or lowest income) and then adds these values, Gini is bigger than any other concentration coefficient that uses rankings that do no rank by the income concept of interest.

Table 2. Marginal Contribution of a Tax without Another Tax or Transfer in Place

Adding a Tax that is		
Regressive $\Pi_T^K < 0$	Neutral $\Pi_T^K = 0$	Progressive $\Pi_T^K > 0$
Always unequalizing	Always no change in equality or unequalizing	Equalizing if and only if equation 22 holds

Table 3 shows that adding a neutral tax (where progressivity is calculated with respect to households ranked by the original income) could be unequalizing.

Table 3. Addition of a Neutral Tax with an Unequalizing Effect

Individual	**Original Income (OI)**	**Tax (T)**	**OI–T**
1	1.00	0.00	1.00
2	11.00	0.00	11.00
3	12.00	10.00	2.00
4	13.00	0.00	13.00
Total	**37.00**	**10.00**	**27.00**
Average	**9.25**	**2.50**	**6.75**
Gini	**0.2500**	**n.c.**	**0.4167**
$\boldsymbol{C^X}$	**n.c.**	**0.0000**	**n.c.**
$\Pi_T^{K^X}$	**…**	**0.0000**	**…**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
… Not applicable.

3.2. Adding a Tax to a System that has a Transfer in Place

Because there is only one transfer in place and only one tax is added, equation 13 can be simplified as follows:

$$\boldsymbol{M_T} = \left\{\left(\boldsymbol{G_X} - \boldsymbol{C_X^{X-T+B}}\right) + \left(\frac{(\boldsymbol{1-g})\Pi_T^{RS^{X-T+B}} + (\boldsymbol{1+b})\boldsymbol{\rho_B^{RS^{X-T+B}}}}{\boldsymbol{1-g+b}}\right)\right\} - \left\{\left(\boldsymbol{G_X} - \boldsymbol{C_X^{X+B}}\right) + \boldsymbol{\rho_B^{RS^{X+B}}}\right\}.$$

The preceding equation can be simplified one more step, as

$$\boldsymbol{M_T} = \left(\boldsymbol{C_X^{X+B}} - \boldsymbol{C_X^{X-T+B}}\right) + \left(\frac{(\boldsymbol{1}-\boldsymbol{g})\Pi_T^{RS^{X-T+B}} + (\boldsymbol{1}+\boldsymbol{b})\boldsymbol{\rho_B^{RS^{X-T+B}}}}{\boldsymbol{1}-\boldsymbol{g}+\boldsymbol{b}}\right) - \boldsymbol{\rho_B^{RS^{X+B}}}$$

or

$$\boldsymbol{M_T} = \left(\boldsymbol{C_X^{X+B}} - \boldsymbol{C_X^{X-T+B}}\right) + \left(\frac{(\boldsymbol{1}-\boldsymbol{g})\Pi_T^{RS^{X-T+B}} + \boldsymbol{g}\boldsymbol{\rho_B^{RS^{X-T+B}}}}{\boldsymbol{1}-\boldsymbol{g}+\boldsymbol{b}}\right) + \left(\boldsymbol{\rho_B^{RS^{X-T+B}}} - \boldsymbol{\rho_B^{RS^{X+B}}}\right).$$

Recalling the notation section and the definitions of $\boldsymbol{\rho_B^{RS^{X-T+B}}}$ and $\boldsymbol{\rho_B^{RS^{X+B}}}$ which are equal to $(\boldsymbol{C_X^{X-T+B}} - \boldsymbol{C_{X+B}^{X-T+B}})$ and $(\boldsymbol{C_X^{X+B}} - \boldsymbol{G_{X+B}})$ respectively, we can rewrite the preceding equation as follows:

(23) $$\boldsymbol{M_T} = \left(\frac{(\boldsymbol{1}-\boldsymbol{g})\Pi_T^{RS^{X-T+B}} + \boldsymbol{g}\boldsymbol{\rho_B^{RS^{X-T+B}}}}{\boldsymbol{1}-\boldsymbol{g}+\boldsymbol{b}}\right) + \left(\boldsymbol{G_{X+B}} - \boldsymbol{C_{X+B}^{X-T+B}}\right).$$

Now, notice that based on equation 23, if ranking of the households does not change before and after adding the tax, the last parentheses become equal to zero. As discussed previously, the last set of parentheses is generally a non-negative term and reaches its minimum when ranking of individuals before and after adding the tax remains the same.

Now, using these generally defined Kakwani indexes, equation 23 can be written as follows:

(24) $$\boldsymbol{M_T} = \left(\frac{\boldsymbol{g}\Pi_T^{K^{X-T+B}} + \frac{\boldsymbol{gb}}{\boldsymbol{1+b}}\boldsymbol{\rho_B^{K^{X-T+B}}}}{\boldsymbol{1}-\boldsymbol{g}+\boldsymbol{b}}\right) + \left(\boldsymbol{G_{X+B}} - \boldsymbol{C_{X+B}^{X-T+B}}\right).$$

For a tax to be equalizing, equation (3.2.2) should be positive, that is,

(25)

$$\boldsymbol{M_T} = \left(\frac{\boldsymbol{g}\Pi_T^{K^{X-T+B}} + \frac{\boldsymbol{gb}}{\boldsymbol{1+b}}\boldsymbol{\rho_B^{K^{X-T+B}}}}{\boldsymbol{1}-\boldsymbol{g}+\boldsymbol{b}}\right) + \left(\boldsymbol{G_{X+B}} - \boldsymbol{C_{X+B}^{X-T+B}}\right) > 0.$$

Using the preceding condition, table 4 helps to determine whether adding a tax to a system with a transfer in place would reduce inequality.

Table 4. Marginal Contribution of a Tax with a Transfer in Place

		To a system with a Transfer that with respect to the end income ranking is		
		Regressive $\rho_B^{K^{X-T+B}} < 0$	Neutral $\rho_B^{K^{X-T+B}} = 0$	Progressive $\rho_B^{K^{X-T+B}} > 0$
Adding a Tax that with respect to the end income ranking is	Regressive $\Pi_T^{K^{X-T+B}} < 0$	More equalizing if and only if condition 25 holds	More equalizing if and only if condition 25 holds	More equalizing if and only if condition 25 holds
	Neutral $\Pi_T^{K^{X-T+B}} = 0$	More equalizing if and only if condition 25 holds	More equalizing if and only if condition 25 holds	Always equalizing
	Progressive $\Pi_T^{K^{X-T+B}} > 0$	More equalizing if and only if condition 25 holds	Always equalizing	Always equalizing

In calculating progressivity, households' rank with respect to their original income is used.

The most counterintuitive result is that adding a regressive tax to a regressive transfer, where progressivity is calculated with respect to the final income ranking of households, can reduce inequality. The following examples illustrate this case and other counterintuitive results.

Table 5 shows that adding a regressive tax to a fiscal system with a regressive transfer (where progressivity is calculated with respect to households ranked by original income) could be equalizing.

Table 5. Addition of a Regressive Tax with an Equalizing Effect to a Fiscal System with a Regressive Transfer

Individual	**Original Income (OI)**	**Benefit (B)**	**OI + B**	**Tax (T)**	**OI – T**	**End Income (EI)**
1	10.00	0.90	10.90	1.00	9.00	9.90
2	11.00	0.00	11.00	0.20	10.80	10.80
3	12.00	1.10	13.10	2.20	9.80	10.90
4	13.00	0.00	13.00	4.10	8.90	8.90
Total	**46.00**	**2.00**	**48.00**	**7.50**	**38.50**	**40.50**
Average	**11.50**	**0.50**	**12.00**	**1.88**	**9.63**	**10.13**
Gini	**0.0543**	**n.c.**	**0.0448**	**n.c.**	**0.0422**	**0.0426**
C^{X-T+B}	**−0.0109**	**0.3**	**0.0021**	**−0.2167**	**0.0292**	**n.c.**
$\Pi_T^{K^{X-T+B}}$ **or** $\rho_B^{K^{X-T+B}}$	**…**	**−0.3109**	**…**	**−0.2058**	**…**	**…**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
… Not applicable.

Table 6 shows that adding a regressive tax to a fiscal system with a neutral transfer (where progressivity is calculated with respect to households ranked by original income) could be equalizing.

Table 6. Addition of a Regressive Tax with an Equalizing Effect to a Fiscal System with a Neutral Transfer

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.00	1.00	0.00	1.00	1.00
2	11.00	0.00	11.00	1.10	9.90	9.90
3	12.00	2.00	14.00	3.00	9.00	11.00
4	13.00	0.00	13.00	1.00	12.00	12.00
Total	**37.00**	**2.00**	**39.00**	**5.10**	**31.90**	**33.90**
Average	**9.25**	**0.50**	**9.75**	**1.28**	**7.98**	**8.48**
Gini	**0.2500**	**n.c.**	**0.2628**	**n.c.**	**0.2657**	**0.2515**
C^{X-T+B}	**0.2500**	**0.2500**	**0.2500**	**0.2402**	**0.2516**	**n.c.**
$\Pi_T^{K^{X-T+B}}$ or $\rho_B^{K^{X-T+B}}$	**…**	**0.0000**	**…**	**-0.0098**	**…**	**…**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
… Not applicable.

Table 7 shows that adding a regressive tax to a fiscal system with a progressive transfer (where progressivity is calculated with respect to households ranked by original income) could be equalizing.

Table 7. Addition of a Regressive Tax with an Equalizing Effect to a Fiscal System with a Progressive Transfer

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.10	1.10	0.00	1.00	1.10
2	11.00	0.00	11.00	1.10	9.90	9.90
3	12.00	1.10	13.10	3.00	9.00	10.10
4	13.00	0.00	13.00	1.00	12.00	12.00
Total	**37.00**	**1.20**	**38.20**	**5.10**	**31.90**	**33.10**
Average	**9.25**	**0.30**	**9.55**	**1.28**	**7.98**	**8.28**
Gini	**0.2500**	**n.c.**	**0.2487**	**n.c.**	**0.2657**	**0.2485**
C^{X-T+B}	**0.2500**	**0.1667**	**0.2474**	**0.2402**	**0.2516**	**n.c.**
$\Pi_T^{K^{X-T+B}}$ or $\rho_B^{K^{X-T+B}}$	**…**	**0.0833**	**…**	**−0.0098**	**…**	**…**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated. / … Not applicable.

Table 8 shows that adding a neutral tax to a fiscal system with a regressive transfer (where progressivity is calculated with respect to households ranked by original income) could be equalizing.

Table 8. Addition of a Neutral Tax with an Equalizing Effect to a Fiscal System with a Regressive Transfer

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.00	1.00	0.00	1.00	1.00
2	11.00	0.00	11.00	1.00	10.00	10.00
3	12.00	2.00	14.00	3.00	9.00	11.00
4	13.00	0.10	13.10	1.00	12.00	12.10
Total	**37.00**	**2.10**	**39.10**	**5.00**	**32.00**	**34.10**
Average	**9.25**	**0.53**	**9.78**	**1.25**	**8.00**	**8.53**
Gini	**0.2500**	**n.c.**	**0.2628**	**n.c.**	**0.2656**	**0.2515**
C^{X-T+B}	**0.2500**	**0.2738**	**0.2513**	**0.2500**	**0.2500**	**n.c.**
$\Pi_T^{K^{X-T+B}}$ or $\rho_B^{K^{X-T+B}}$	**…**	**–0.0238**	**…**	**0.0000**	**…**	**…**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
… Not applicable.

Table 9 shows that adding a neutral tax to a fiscal system with a neutral transfer (where progressivity is calculated with respect to households ranked by original income) could be equalizing.

Table 9. Addition of a Neutral Tax with an Equalizing Effect to a Fiscal System with a Neutral Transfer

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.00	1.00	0.00	1.00	1.00
2	11.00	0.00	11.00	2.00	9.00	9.00
3	12.00	2.00	14.00	4.00	8.00	10.00
4	13.00	0.00	13.00	2.00	11.00	11.00
Total	**37.00**	**2.00**	**39.00**	**8.00**	**29.00**	**31.00**
Average	**9.25**	**0.50**	**9.75**	**2.00**	**7.25**	**7.75**
Gini	**0.2500**	**n.c.**	**0.2628**	**n.c.**	**0.2672**	**0.2500**
C^{X-T+B}	**0.2500**	**0.2500**	**0.2500**	**0.2500**	**0.2500**	**n.c.**
$\Pi_T^{K^{X-T+B}}$ or $\rho_B^{K^{X-T+B}}$	**…**	**0.0000**	**…**	**0.0000**	**…**	**…**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
… Not applicable.

Table 10 shows that adding a neutral tax to a fiscal system with a progressive transfer (where progressivity is calculated with respect to households ranked by original income) could be equalizing.

Table 10. Addition of a Neutral Tax with an Equalizing Effect to a Fiscal System with a Progressive Transfer

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.10	1.10	0.00	1.00	1.10
2	11.00	0.00	11.00	1.00	10.00	10.00
3	12.00	2.00	14.00	3.00	9.00	11.00
4	13.00	0.00	13.00	1.00	12.00	12.00
Total	**37.00**	**2.10**	**39.10**	**5.00**	**32.00**	**34.10**
Average	**9.25**	**0.53**	**9.78**	**1.25**	**8.00**	**8.53**
Gini	**0.2500**	**n.c.**	**0.2602**	**n.c.**	**0.2656**	**0.2471**
C^{X-T+B}	**0.2500**	**0.2024**	**0.2474**	**0.2500**	**0.2500**	**n.c.**
$\Pi_T^{K^{X-T+B}}$ **or** $\rho_B^{K^{X-T+B}}$	**…**	**0.0476**	**…**	**0.0000**	**…**	**…**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
… Not applicable.

Table 11 shows that adding a progressive tax to a fiscal system with a regressive transfer (where progressivity is calculated with respect to households ranked by original income) could be unequalizing.

Table 11. Addition of a Progressive Tax with an Unequalizing Effect to a Fiscal System with a Regressive Transfer

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.00	1.00	0.00	1.00	1.00
2	11.00	0.00	11.00	0.10	10.90	10.90
3	12.00	0.00	12.00	1.00	11.00	11.00
4	13.00	4.00	17.00	0.20	12.80	16.80
Total	**37.00**	**4.00**	**41.00**	**1.30**	**35.70**	**39.70**
Average	**9.25**	**1.00**	**10.25**	**0.33**	**8.93**	**9.93**
Gini	**0.2500**	**n.c.**	**0.2988**	**n.c.**	**0.2486**	**0.2991**
C^{X-T+B}	**0.2500**	**0.7500**	**0.2988**	**0.2885**	**0.2486**	**n.c.**
$\Pi_T^{K^{X-T+B}}$ **or** $\rho_B^{K^{X-T+B}}$	**…**	**−0.5000**	**…**	**0.0385**	**…**	**…**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
… Not applicable.

Although equation 23 is derived using the R-S index that is calculated with respect to the end income ranking of households, one can calculate a similar derivation using the R-S index that is calculated with respect to the original income ranking, as shown in the following equation:

$$M_T = \left(\frac{(1-g)\Pi_T^{RS} + g\rho_B^{RS}}{1-g+b}\right) + \left[\underbrace{\left(C_{X-T+B}^{X} - G_{X-T+B}\right)}_{Reranking\ in\ the\ whole\ system} - \underbrace{\left(C_{X+B}^{X} - G_{X+B}\right)}_{Reranking\ before\ the\ tax\ is\ added}\right].$$

Because both terms in the brackets are non-positive, the bracket could be positive, zero, or negative. For the tax to be equalizing, the following condition should hold:

$$\left(\frac{(1-g)\Pi_T^{RS} + g\rho_B^{RS}}{1-g+b}\right) + \left[\left(C_{X-T+B}^{X} - G_{X-T+B}\right) - \left(C_{X+B}^{X} - G_{X+B}\right)\right] > 0$$

or

(26)

$$\left(\frac{g\Pi_T^{K} + \frac{gb}{1+b}\rho_B^{K}}{1-g+b}\right) + \overbrace{\left[\underbrace{\left(\boldsymbol{C_{X-T+B}^{X} - G_{X-T+B}}\right)}_{\boldsymbol{Reranking\ after\ the\ tax\ is\ added}} - \underbrace{\left(\boldsymbol{C_{X+B}^{X} - G_{X+B}}\right)}_{\boldsymbol{Reranking\ before\ the\ tax\ is\ added}}\right]}^{\boldsymbol{Marginal\ effect\ of\ the\ tax\ on\ reranking}} > 0.$$

As shown in table 12, using the traditional Kakwani index (that is, when the index is calculated with respect to the original income ranking of households) would not result in any certainty about whether the addition of a tax reduces inequality.

Table 12. Marginal Contribution of a Tax with a Transfer in Place

		To a system with a Transfer that with respect to the original income ranking is		
		Regressive $\rho_B^K < 0$	Neutral $\rho_B^K = 0$	Progressive $\rho_B^K > 0$
Adding a Tax that with respect to the original income ranking is	Regressive $\Pi_T^K < 0$	Equalizing if and only if condition 26 holds	Equalizing if and only if condition 26 holds	Equalizing if and only if condition 26 holds
	Neutral $\Pi_T^K = 0$	Equalizing if and only if condition 26 holds	Equalizing if and only if condition 26 holds	Equalizing if and only if condition 26 holds
	Progressive $\Pi_T^K > 0$	Equalizing if and only if condition 26 holds	Equalizing if and only if condition 26 holds	Equalizing if and only if condition 26 holds

In calculating progressivity, households' rank with respect to their original income is used.

Table 12 contains some counterintuitive cases that the following examples will help to explain. Table 13, for instance, shows that adding a regressive tax to a fiscal system with a regressive transfer (where progressivity is calculated with respect to households ranked by original income) could be equalizing.

Table 13. Addition of a Regressive Tax with an Equalizing Effect to a Fiscal System with a Regressive Transfer

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.00	1.00	0.10	0.90	0.90
2	11.00	0.00	11.00	1.00	10.00	10.00
3	12.00	2.00	14.00	3.00	9.00	11.00
4	13.00	0.40	13.40	1.00	12.00	12.40
Total	**37.00**	**2.40**	**39.40**	**5.10**	**31.90**	**34.30**
Average	**9.25**	**0.60**	**9.85**	**1.28**	**7.98**	**8.58**
Gini	**0.2500**	**n.c.**	**0.2627**	**n.c.**	**0.2688**	**0.2587**
C^X	**0.2500**	**0.3333**	**0.2551**	**0.2304**	**0.2531**	**0.2587**
Π_T^K or ρ_B^K	**…**	**−0.0051**	**…**	**−0.0031**	**…**	**…**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
… Not applicable.

Table 14 shows that adding a regressive tax to a fiscal system with a neutral transfer (where progressivity is calculated with respect to households ranked by original income) could be equalizing.

Table 14. Addition of a Regressive Tax with an Equalizing Effect to a Fiscal System with a Neutral Transfer

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.00	1.00	0.10	0.90	0.90
2	11.00	0.00	11.00	1.00	10.00	10.00
3	12.00	2.00	14.00	3.00	9.00	11.00
4	13.00	0.00	13.00	1.00	12.00	12.00
Total	**37.00**	**2.00**	**39.00**	**5.10**	**31.90**	**33.90**
Average	**9.25**	**0.50**	**9.75**	**1.28**	**7.98**	**8.48**
Gini	**0.2500**	**n.c.**	**0.2628**	**n.c.**	**0.2688**	**0.2529**
C^X	**0.2500**	**0.2500**	**0.2500**	**0.2304**	**0.2531**	**0.2529**
Π_T^K or ρ_B^K	**…**	**0.0000**	**…**	**−0.0031**	**…**	**…**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
… Not applicable.

Table 15 shows that adding a regressive tax to a fiscal system with a progressive transfer (where progressivity is calculated with respect to households ranked by original income) could be equalizing.

Table 15. Addition of a Regressive Tax with an Equalizing Effect to a Fiscal System with a Progressive Transfer

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	1.00	2.00	0.10	0.90	1.90
2	11.00	0.00	11.00	1.00	10.00	10.00
3	12.00	2.00	14.00	3.00	9.00	11.00
4	13.00	0.40	13.40	1.00	12.00	12.40
Total	**37.00**	**3.40**	**40.40**	**5.10**	**31.90**	**35.30**
Average	**9.25**	**0.85**	**10.10**	**1.28**	**7.98**	**8.83**
Gini	**0.2500**	**n.c.**	**0.2376**	**n.c.**	**0.2688**	**0.2302**
C^X	**0.2500**	**0.0147**	**0.2302**	**0.2304**	**0.2531**	**0.2302**
Π_T^K or ρ_B^K	…	**0.0198**	…	**−0.0031**	…	…

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
… Not applicable.

Table 16 shows that adding a neutral tax to a fiscal system with a regressive transfer (where progressivity is calculated with respect to households ranked by original income) could be equalizing.

Table 16. Addition of a Neutral Tax with an Equalizing Effect to a Fiscal System with a Regressive Transfer

Individual	Original Income (OI)	Benefit (B)	OI+B	Tax (T)	OI−T	End Income (EI)
1	1.00	0.00	1.00	0.00	1.00	1.00
2	11.00	0.00	11.00	1.00	10.00	10.00
3	12.00	2.00	14.00	3.00	9.00	11.00
4	13.00	0.10	13.10	1.00	12.00	12.10
Total	**37.00**	**2.10**	**39.10**	**5.00**	**32.00**	**34.10**
Average	**9.25**	**0.53**	**9.78**	**1.25**	**8.00**	**8.53**
Gini	**0.2500**	**n.c.**	**0.2628**	**n.c.**	**0.2656**	**0.2515**
C^X	**0.2500**	**0.2738**	**0.2513**	**0.2500**	**0.2500**	**0.2515**
Π_T^K or ρ_B^K	…	**−0.0013**	…	**0.0000**	…	…

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
… Not applicable.

Table 17 shows that adding a neutral tax to a fiscal system with a regressive transfer (where progressivity is calculated with respect to households ranked by original income) could be unequalizing.

Table 17. Addition of a Neutral Tax with an Unequalizing Effect to a Fiscal System with a Regressive Transfer

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.00	1.00	0.00	1.00	1.00
2	11.00	0.00	11.00	1.00	10.00	10.00
3	12.00	2.00	14.00	5.00	7.00	9.00
4	13.00	0.10	13.10	1.00	12.00	12.10
Total	**37.00**	**2.10**	**39.10**	**7.00**	**30.00**	**32.10**
Average	**9.25**	**0.53**	**9.78**	**1.75**	**7.50**	**8.03**
Gini	**0.2500**	**n.c.**	**0.2628**	**n.c.**	**0.3000**	**0.2671**
C^X	**0.2500**	**0.2738**	**0.2513**	**0.2500**	**0.2500**	**0.2516**
Π_T^K or ρ_B^K	…	**−0.0013**	…	**0.0000**	…	…

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
… Not applicable.

Table 18 shows that adding a neutral tax to a fiscal system with a neutral transfer (where progressivity is calculated with respect to households ranked by original income) could be unequalizing.

Table 18. Addition of a Neutral Tax with an Unequalizing Effect to a Fiscal System with a Neutral Transfer

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.00	1.00	0.00	1.00	1.00
2	11.00	0.00	11.00	1.00	10.00	10.00
3	12.00	2.00	14.00	5.00	7.00	9.00
4	13.00	0.00	13.00	1.00	12.00	12.00
Total	**37.00**	**2.00**	**39.00**	**7.00**	**30.00**	**32.00**
Average	**9.25**	**0.50**	**9.75**	**1.75**	**7.50**	**8.00**
Gini	**0.2500**	**n.c.**	**0.2628**	**n.c.**	**0.3000**	**0.2656**
C^X	**0.2500**	**0.2500**	**0.2500**	**0.2500**	**0.2500**	**0.2500**
Π_T^K or ρ_B^K	…	**0.0000**	…	**0.0000**	…	…

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
… Not applicable.

Table 19 shows that adding a neutral tax to a fiscal system with a neutral transfer (where progressivity is calculated with respect to households ranked by original income) could be equalizing.

Table 19. Addition of a Neutral Tax with an Equalizing Effect to a Fiscal System with a Neutral Transfer

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.00	1.00	0.00	1.00	1.00
2	11.00	0.00	11.00	1.00	10.00	10.00
3	12.00	2.00	14.00	3.00	9.00	11.00
4	13.00	0.00	13.00	1.00	12.00	12.00
Total	**37.00**	**2.00**	**39.00**	**5.00**	**32.00**	**34.00**
Average	**9.25**	**0.50**	**9.75**	**1.25**	**8.00**	**8.50**
Gini	**0.2500**	**n.c.**	**0.2628**	**n.c.**	**0.2656**	**0.2500**
C^X	**0.2500**	**0.2500**	**0.2500**	**0.2500**	**0.2500**	**0.2500**
Π_T^K or ρ_B^K	**…**	**0.0000**	**…**	**0.0000**	**…**	**…**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
… Not applicable.

Table 20 shows that adding a neutral tax to a fiscal system with a progressive transfer (where progressivity is calculated with respect to households ranked by original income) could be equalizing, where progressivity is calculated with respect to the original income ranking of households.

Table 20. Addition of a Neutral Tax with an Equalizing Effect to a Fiscal System with a Progressive Transfer

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.00	1.00	0.00	1.00	1.00
2	11.00	0.10	11.10	1.00	10.00	10.10
3	12.00	2.00	14.00	3.00	9.00	11.00
4	13.00	0.00	13.00	1.00	12.00	12.00
Total	**37.00**	**2.10**	**39.10**	**5.00**	**32.00**	**34.10**
Average	**9.25**	**0.53**	**9.78**	**1.25**	**8.00**	**8.53**
Gini	**0.2500**	**n.c.**	**0.2615**	**n.c.**	**0.2656**	**0.2485**
C^X	**0.2500**	**0.2262**	**0.2487**	**0.2500**	**0.2500**	**0.2485**
Π_T^K or ρ_B^K	**…**	**0.0013**	**…**	**0.0000**	**…**	**…**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
… Not applicable.

Table 21 shows that adding a neutral tax to a fiscal system with a progressive transfer (where progressivity is calculated with respect to households ranked by original income) could be unequalizing.

Table 21. Addition of a Neutral Tax with an Unequalizing Effect to a Fiscal System with a Progressive Transfer

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.00	1.00	0.00	1.00	1.00
2	11.00	0.10	11.10	1.00	10.00	10.10
3	12.00	2.00	14.00	5.00	7.00	9.00
4	13.00	0.00	13.00	1.00	12.00	12.00
Total	**37.00**	**2.10**	**39.10**	**7.00**	**30.00**	**32.10**
Average	**9.25**	**0.53**	**9.78**	**1.75**	**7.50**	**8.03**
Gini	**0.2500**	**n.c.**	**0.2615**	**n.c.**	**0.3000**	**0.2656**
C^X	**0.2500**	**0.2262**	**0.2487**	**0.2500**	**0.2500**	**0.2484**
Π_T^K or ρ_B^K	…	**0.0013**	…	**0.0000**	…	…

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
… Not applicable.

Table 22 shows that adding a progressive tax to a fiscal system with a regressive transfer (where progressivity is calculated with respect to households ranked by original income) could be unequalizing.

Table 22. Addition of a Progressive Tax with an Unequalizing Effect to a Fiscal System with a Regressive Transfer

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.00	1.00	0.00	1.00	1.00
2	11.00	0.00	11.00	1.00	10.00	10.00
3	12.00	2.00	14.00	5.00	7.00	9.00
4	13.00	0.10	13.10	1.10	11.90	12.00
Total	**37.00**	**2.10**	**39.10**	**7.10**	**29.90**	**32.00**
Average	**9.25**	**0.53**	**9.78**	**1.78**	**7.48**	**8.00**
Gini	**0.2500**	**n.c.**	**0.2628**	**n.c.**	**0.2985**	**0.2656**
C^X	**0.2500**	**0.2738**	**0.2513**	**0.2570**	**0.2483**	**0.2500**
Π_T^K or ρ_B^K	…	**−0.0013**	…	**0.0017**	…	…

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
… Not applicable.

Table 23 shows that adding a progressive tax to a fiscal system with a neutral transfer (where progressivity is calculated with respect to households ranked by original income) could be unequalizing.

Table 23. Addition of a Progressive Tax with an Unequalizing Effect to a Fiscal System with a Neutral Transfer

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	10.00	1.00	11.00	0.00	10.00	11.00
2	11.00	1.10	12.10	0.00	11.00	12.10
3	12.00	1.20	13.20	0.00	12.00	13.20
4	13.00	1.30	14.30	5.00	8.00	9.30
Total	**46.00**	**4.60**	**50.60**	**5.00**	**41.00**	**45.60**
Average	**11.50**	**1.15**	**12.65**	**1.25**	**10.25**	**11.40**
Gini	**0.0543**	**n.c.**	**0.0543**	**n.c.**	**0.0793**	**0.0702**
C^X	**0.0543**	**0.0543**	**0.0543**	**0.7500**	**-0.0305**	**-0.0219**
Π_T^K or ρ_B^K	…	**0.0000**	…	**0.0848**	…	…

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
… Not applicable.

Table 24 shows that adding a progressive tax to a fiscal system with a progressive transfer (where progressivity is calculated with respect to households ranked by original income) could be unequalizing.

Table 24. Addition of a Progressive Tax with an Unequalizing Effect to a Fiscal System with a Progressive Transfer

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	10.00	0.10	1.10	0.00	1.00	1.10
2	11.00	0.00	11.00	1.00	10.00	10.00
3	12.00	2.00	14.00	5.00	7.00	9.00
4	13.00	0.00	13.00	1.10	11.90	11.90
Total	**46.00**	**2.10**	**39.10**	**7.10**	**29.90**	**32.00**
Average	**11.50**	**0.53**	**9.78**	**1.78**	**7.48**	**8.00**
Gini	**0.0543**	**n.c.**	**0.2602**	**n.c.**	**0.2985**	**0.2609**
C^X	**0.0543**	**0.2024**	**0.2474**	**0.2570**	**0.2483**	**0.2453**
Π_T^K or ρ_B^K	…	**0.0026**	…	**0.0017**	…	…

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
… Not applicable.

3.3. Adding a Tax to a System with Multiple Taxes and Transfers in Place

Recall from equation 13 that[10]

$$\boldsymbol{M_{T_1}} = \left\{\left(\boldsymbol{G_X} - \boldsymbol{C_X^Z}\right) + \left(\frac{\sum_{i=1}^{n}(1-g_i)\Pi_{T_i}^{RS^Z} + \sum_{j=1}^{m}(1+b_j)\rho_{B_j}^{RS^Z}}{1-\sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j}\right)\right\} - \left\{\left(\boldsymbol{G_X} - \boldsymbol{C_X^{Z\setminus T_1}}\right) + \left(\frac{\sum_{i=2}^{n}(1-g_i)\Pi_{T_i}^{RS^{Z\setminus T_1}} + \sum_{j=1}^{m}(1+b_j)\rho_{B_j}^{RS^{Z\setminus T_1}}}{1-\sum_{i=2}^{n} g_i + \sum_{j=1}^{m} b_j}\right)\right\}.$$

For T_1 to be equalizing, this equation has to be positive, that is,

(27)

$$\left\{\left(\boldsymbol{G_X} - \boldsymbol{C_X^Z}\right) + \left(\frac{\sum_{i=1}^{n}(1-g_i)\Pi_{T_i}^{RS^Z} + \sum_{j=1}^{m}(1+b_j)\rho_{B_j}^{RS^Z}}{1-\sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j}\right)\right\} - \left\{\left(\boldsymbol{G_X} - \boldsymbol{C_X^{Z\setminus T_1}}\right) + \left(\frac{\sum_{i=2}^{n}(1-g_i)\Pi_{T_i}^{RS^{Z\setminus T_1}} + \sum_{j=1}^{m}(1+b_j)\rho_{B_j}^{RS^{Z\setminus T_1}}}{1-\sum_{i=2}^{n} g_i + \sum_{j=1}^{m} b_j}\right)\right\} > 0.$$

If adding this specific tax does not change the end income ranking of households (that is, if end income rankings are the same before and after adding the tax), then ranking with respect to Z and Y is the same, which simplifies the whole equation to

$$\left(1 - \sum_{i=2}^{n} g_i + \sum_{j=1}^{m} b_j\right)(1-g_1)\Pi_{T_1}^{RS^Z} > -g_1\left(\sum_{i=2}^{n}(1-g_i)\Pi_{T_i}^{RS^{Z\setminus T_1}} + \sum_{j=1}^{m}(1+b_j)\rho_{B_j}^{RS^{Z\setminus T_1}}\right)$$

which is equal to

$$\Pi_{T_1}^{RS^Z} > -\frac{g_1}{(1-g_1)}\left(\frac{\sum_{i=2}^{n}(1-g_i)\Pi_{T_i}^{RS^{Z\setminus T_1}} + \sum_{j=1}^{m}(1+b_j)\rho_{B_j}^{RS^{Z\setminus T_1}}}{1-\sum_{i=2}^{n} g_i + \sum_{j=1}^{m} b_j}\right)$$

or

$$\Pi_{T_1}^{RS^Z} > -\frac{g_1}{(1-g_1)}\left(C_X^{\boldsymbol{Z\setminus T_1}} - G_{\boldsymbol{Z\setminus T_1}}\right)$$

or

[10] Recall from the notation section that $Z = X - \sum_{i=1}^{n} T_i + \sum_{j=1}^{m} B_j$ and $Z\setminus T_1 = X - \sum_{i=2}^{n} T_i + \sum_{j=1}^{m} B_j$.

(28)

$$\Pi_{\boldsymbol{T_1}}^{\boldsymbol{K^Z}} < (\boldsymbol{C_X^{Z\backslash T_1}} - \boldsymbol{G_{Z\backslash T_1}}).$$

The term on the right-hand side is the modified VE term, which was introduced in the notation section as

$$VE_{X,Z\backslash T_1}^{Z\backslash T_1} = C_X^{Z\backslash T_1} - G_{Z\backslash T_1}.$$

Thus, equation 28 can be written as follows:

(29)

$$\Pi_{\boldsymbol{T_1}}^{\boldsymbol{K^Z}} < \boldsymbol{VE_{X,Z\backslash T_1}^{Z\backslash T_1}}.$$

Table 25 shows how one can determine whether adding a tax to a system of taxes and transfers reduces inequality when the new tax does not change the end income ranking of households.

Table 25. Marginal Contribution of a Tax with Multiple Taxes and Transfers in Place

		To a system with multiple taxes and transfers where its vertical equity (with respect to the final income ranking) is		
		Negative $\boldsymbol{VE_{X,Z\backslash T_1}^{Z\backslash T_1}} < 0$	Zero $\boldsymbol{VE_{X,Z\backslash T_1}^{Z\backslash T_1}} = 0$	Positive $\boldsymbol{VE_{X,Z\backslash T_1}^{Z\backslash T_1}} > 0$
Adding a Tax that with respect to the final incomes ranking (Z) is	Regressive $\Pi_T^{K^Z} < 0$	Equalizing if and only if condition 29 holds	Always equalizing	Always equalizing
	Neutral $\Pi_T^{K^Z} = 0$	Always unequalizing	No change in inequality	Always equalizing
	Progressive $\Pi_T^{K^Z} > 0$	Always unequalizing	Always unequalizing	Equalizing if and only if condition 29 holds

$Z = X - \sum_{i=1}^{n} T_i + \sum_{j=1}^{m} B_j$ and $Z\backslash T_1 = X - \sum_{i=2}^{n} T_i + \sum_{j=1}^{m} B_j$. The new tax does not change the end income ranking of individuals.

For the results in table 25 to hold, the tax that we are interested in should not have any effect on the end income ranking of households. If that is not the case, then equation 27 cannot be simplified much further and the effect of adding such a tax cannot be determined using a simple rule of thumb from the table.

Alternatively, one can use the progressivity with respect to the original income in the analysis. For this purpose, we need to use equation 16:

$$\boldsymbol{M_{T_1}} = \left[\left(\frac{\left[(\mathbf{1}-\boldsymbol{g_1})\Pi_{T_1}^{RS}\right]+\left[(\boldsymbol{g_1})\overbrace{\left(\boldsymbol{G_X}-\boldsymbol{C_{X\backslash T_1}^X}\right)}^{\boldsymbol{VE\ of\ the\ system\ without\ T_1}}\right]}{\left(\mathbf{1}-\sum_{i=1}^{n}\boldsymbol{g_i}+\sum_{j=1}^{m}\boldsymbol{b_j}\right)}\right)\right] + \underbrace{\left[\left(\boldsymbol{C_Z^X}-\boldsymbol{G_Z}\right)-\left(\boldsymbol{C_{X\backslash T_1}^X}-\boldsymbol{G_{X\backslash T_1}}\right)\right]}_{\boldsymbol{Contribution\ of\ T_1 to\ reranking}}.$$

For a tax to be equalizing when it is added to a system of taxes and transfers, the following condition should hold:

(30)

$$\boldsymbol{M_{T_1}} = \left[\left(\frac{\left[(\mathbf{1}-\boldsymbol{g_1})\Pi_{T_1}^{RS}\right]+\left[(\boldsymbol{g_1})\overbrace{\left(\boldsymbol{G_X}-\boldsymbol{C_{X\backslash T_1}^X}\right)}^{\boldsymbol{VE\ of\ the\ system\ without\ T_1}}\right]}{\left(\mathbf{1}-\sum_{i=1}^{n}\boldsymbol{g_i}+\sum_{j=1}^{m}\boldsymbol{b_j}\right)}\right)\right] + \underbrace{\left[\left(\boldsymbol{C_Z^X}-\boldsymbol{G_Z}\right)-\left(\boldsymbol{C_{X\backslash T_1}^X}-\boldsymbol{G_{X\backslash T_1}}\right)\right]}_{\boldsymbol{Contribution\ of\ T_1 to\ reranking}} > 0$$

or

(31)

$$\boldsymbol{M_{T_1}} = \left[\left(\frac{\left[\boldsymbol{g_1}\Pi_{T_1}^{K}\right]+\left[(\boldsymbol{g_1})\overbrace{\left(\boldsymbol{G_X}-\boldsymbol{C_{X\backslash T_1}^X}\right)}^{\boldsymbol{VE\ of\ the\ system\ without\ T_1}}\right]}{\left(\mathbf{1}-\sum_{i=1}^{n}\boldsymbol{g_i}+\sum_{j=1}^{m}\boldsymbol{b_j}\right)}\right)\right] + \underbrace{\left[\left(\boldsymbol{C_Z^X}-\boldsymbol{G_Z}\right)-\left(\boldsymbol{C_{X\backslash T_1}^X}-\boldsymbol{G_{X\backslash T_1}}\right)\right]}_{\boldsymbol{Contribution\ of\ T_1 to\ reranking}} > 0.$$

4. In the Presence of Reranking, is the Marginal Contribution of a Transfer Equalizing?

This section is similar to the previous one, so I have presented only the minimum derivations except in cases of significant differences.

4.1. The Case of Only One Transfer

Similarly to section 3.1, we begin with the following equation (using equation 18):

(32)

$$M_B = \rho_B^{RS} + (C_{X+B}^X - G_{X+B})$$

For a transfer to be equalizing, equation 32 has to be positive, that is,

$$M_B = \rho_B^{RS} + (C_{X+B}^X - G_{X+B}) > 0$$

or

(33)

$$\rho_B^{RS} > (G_{X+B} - C_{X+B}^{X})$$

or

(34)

$$\boldsymbol{\rho_B^K} > \left(\frac{\boldsymbol{1+b}}{\boldsymbol{b}}\right)(\boldsymbol{G_{X+B}} - \boldsymbol{C_{X+B}^{X}}).$$

As in the previous section, the right-hand side is non-negative and reaches zero if the transfer does not change the ranking of individuals.

Table 26. Marginal Contribution of a Transfer with No Other Tax or Transfer in Place

Adding a Transfer that is		
Regressive $\boldsymbol{\rho_B^K} < 0$	Neutral $\boldsymbol{\rho_B^K} = \boldsymbol{0}$	Progressive $\boldsymbol{\rho_B^K} > 0$
Always unequalizing	Always no change in equality or unequalzing	Equalizing if and only if equation 34 holds

To see how a neutral transfer can be unequalizing in the presence of reranking, refer to table 27.

Table 27. Addition of a Neutral Transfer with Unequalizing Results

Individual	**Original Income (OI)**	**Benefit (B)**	**OI+B**
1	1.00	0.00	1.00
2	11.00	0.00	11.00
3	12.00	10.00	22.00
4	13.00	0.00	13.00
Total	**37.00**	**10.00**	**47.00**
Average	**9.25**	**2.50**	**11.75**
Gini	**0.2500**	**n.c.**	**0.3457**
$\boldsymbol{C^X}$	**n.c.**	**0.0000**	**n.c.**
$\boldsymbol{\rho_B^{K^X}}$	**…**	**0.0000**	**…**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
… Not applicable.

4.2. Adding a Transfer to a System that has a Tax in Place

Because there is only one tax in place and only one transfer is added, equation 14 can be simplified as follows:

$$M_B = \left\{ \left(G_X - C_X^{X-T+B}\right) + \left(\frac{(1-g)\Pi_T^{RS^{X-T+B}} + (1+b)\rho_B^{RS^{X-T+B}}}{1-g+b} \right) \right\} - \left\{ \Pi_T^{RS^{X-T}} + \left(G_X - C_X^{X-T}\right) \right\}.$$

Similarly to section 3.2, this equation can be simplified as follows:

(35)

$$\boldsymbol{M_B = \left(\frac{-b\Pi_T^{RS^{X-T+B}} + (1+b)\rho_B^{RS^{X-T+B}}}{1-g+b} \right) + \left(G_{X-T} - C_{X-T}^{X-T+B} \right)}$$

or

(36)

$$\boldsymbol{M_B = \left(\frac{\frac{-bg}{1-g}\Pi_T^{K^{X-T+B}} + b\rho_B^{K^{X-T+B}}}{1-g+b} \right) + \left(G_{X-T} - C_{X-T}^{X-T+B} \right)}.$$

For a transfer to be equalizing, equation 36 should be positive, that is,

(37)

$$\boldsymbol{M_B = \left(\frac{\frac{-bg}{1-g}\Pi_T^{K^{X-T+B}} + b\rho_B^{K^{X-T+B}}}{1-g+b} \right) + \left(G_{X-T} - C_{X-T}^{X-T+B} \right)} > 0.$$

Using the preceding condition, table 28 helps to determine whether adding a transfer to a system with a tax in place would increase the equality. Note that $\boldsymbol{G_{X-T} - C_{X-T}^{X-T+B}}$ is a non-negative term that reaches zero if adding the benefit does not change the ranking.

Table 28. Marginal Contribution of a Transfer with a Tax in Place

		Adding a Transfer that with respect to the end income ranking is		
		Regressive $\rho_B^{K^{X-T+B}} < 0$	Neutral $\rho_B^{K^{X-T+B}} = 0$	Progressive $\rho_B^{K^{X-T+B}} > 0$
To a system with a Tax that with respect to the end income ranking is	Regressive $\Pi_T^{K^{X-T+B}} < 0$	Equalizing if and only if condition 37 holds	Always equalizing	Always equalizing
	Neutral $\Pi_T^{K^{X-T+B}} = 0$	Equalizing if and only if condition 37 holds	Equalizing if and only if condition 37 holds	Always equalizing
	Progressive $\Pi_T^{K^{X-T+B}} > 0$	Equalizing if and only if condition 37 holds	Equalizing if and only if condition 37 holds	Equalizing if and only if condition 37 holds

In calculating progressivity, households' rank with respect to their original income is used.

Table 28 includes some counterintuitive cases that the following examples will show are indeed possible. Table 29, for instance, shows that adding a regressive transfer to a fiscal system with a regressive tax (where progressivity is calculated with respect to households ranked by original income) could be equalizing.

Table 29. Addition of a Regressive Transfer with an Equalizing Effect to a Fiscal System with a Regressive Tax

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	10.00	2.10	12.10	1.00	9.00	11.10
2	11.00	1.05	12.05	1.00	10.00	11.05
3	12.00	0.00	12.00	1.90	10.10	10.10
4	13.00	0.00	13.00	2.80	10.20	10.20
Total	**46.00**	**3.15**	**49.15**	**6.70**	**39.30**	**42.45**
Average	**11.50**	**0.79**	**12.29**	**1.68**	**9.83**	**10.61**
Gini	**0.0543**	**n.c.**	**0.0155**	**n.c.**	**0.0235**	**0.0227**
C^{X-T+B}	**−0.0435**	**0.5833**	**−0.0033**	**−0.1679**	**−0.0223**	**n.c.**
$\Pi_T^{K^{X-T+B}}$ **or** $\rho_B^{K^{X-T+B}}$	**...**	**−0.6268**	**...**	**−0.1244**	**...**	**...**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
... Not applicable.

Table 30 shows that adding a regressive transfer to a fiscal system with a neutral tax (where progressivity is calculated with respect to households ranked by original income) could be equalizing.

Table 30. Addition of a Regressive Transfer with an Equalizing Effect to a Fiscal System with a Neutral Tax

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.00	1.00	0.00	1.00	1.00
2	11.00	0.00	11.00	1.00	10.00	10.00
3	12.00	2.00	14.00	3.00	9.00	11.00
4	13.00	0.10	13.10	1.00	12.00	12.10
Total	**37.00**	**2.10**	**39.10**	**5.00**	**32.00**	**34.10**
Average	**9.25**	**0.53**	**9.78**	**1.25**	**8.00**	**8.53**
Gini	**0.2500**	**n.c.**	**0.2628**	**n.c.**	**0.2656**	**0.2515**
C^{X-T+B}	**0.2500**	**0.2738**	**0.2513**	**0.2500**	**0.2500**	**n.c.**
$\Pi_T^{K^{X-T+B}}$ **or** $\rho_B^{K^{X-T+B}}$	**...**	**−0.0238**	**...**	**0.0000**	**...**	**...**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
... Not applicable.

Table 31 shows that adding a neutral transfer to a fiscal system with a neutral tax (where progressivity is calculated with respect to households ranked by original income) could be equalizing.

Table 31. Addition of a Neutral Transfer with an Equalizing Effect to a Fiscal System with a Neutral Tax

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI − T	End Income (EI)
1	1.00	0.00	1.00	0.00	1.00	1.00
2	11.00	0.00	11.00	1.00	10.00	10.00
3	12.00	2.00	14.00	3.00	9.00	11.00
4	13.00	0.00	13.00	1.00	12.00	12.00
Total	**37.00**	**2.00**	**39.00**	**5.00**	**32.00**	**34.00**
Average	**9.25**	**0.50**	**9.75**	**1.25**	**8.00**	**8.50**
Gini	**0.2500**	**n.c.**	**0.2628**	**n.c.**	**0.2656**	**0.2500**
C^{X-T+B}	**0.2500**	**0.2500**	**0.2500**	**0.2500**	**0.2500**	**n.c.**
$\Pi_T^{K^{X-T+B}}$ **or** $\rho_B^{K^{X-T+B}}$	**...**	**0.0000**	**...**	**0.0000**	**...**	**...**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
... Not applicable.

Table 32 shows that adding a regressive transfer to a fiscal system with a progressive tax (where progressivity is calculated with respect to households ranked by original income) could be equalizing.

Table 32. Addition of a Regressive Transfer with an Equalizing Effect to a Fiscal System with a Progressive Tax

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI − T	End Income (EI)
1	1.00	0.00	1.00	0.00	1.00	1.00
2	11.00	0.00	11.00	1.00	10.00	10.00
3	12.00	2.00	14.00	3.00	9.00	11.00
4	13.00	0.10	13.10	1.10	11.90	12.00
Total	**37.00**	**2.10**	**39.10**	**5.10**	**31.90**	**34.00**
Average	**9.25**	**0.53**	**9.78**	**1.28**	**7.98**	**8.50**
Gini	**0.2500**	**n.c.**	**0.2628**	**n.c.**	**0.2641**	**0.2500**
C^{X-T+B}	**0.2500**	**0.2738**	**0.2513**	**0.2598**	**0.2484**	**n.c.**
$\Pi_T^{K^{X-T+B}}$ **or** $\rho_B^{K^{X-T+B}}$	**...**	**−0.0238**	**...**	**0.0098**	**...**	**...**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
... Not applicable.

Table 33 shows that adding a neutral transfer to a fiscal system with a progressive tax (where progressivity is calculated with respect to households ranked by original income) could be equalizing.

Table 33. Addition of a Neutral Transfer with an Equalizing Effect to a Fiscal System with a Progressive Tax

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	10.00	11.00	21.00	1.20	8.80	19.80
2	11.00	12.10	23.10	0.00	11.00	23.10
3	12.00	13.20	25.20	0.00	12.00	25.20
4	13.00	14.30	27.30	1.90	11.10	25.40
Total	**46.00**	**50.60**	**96.60**	**3.10**	**42.90**	**93.50**
Average	**11.50**	**12.65**	**24.15**	**0.78**	**10.73**	**23.38**
Gini	**0.0543**	**n.c.**	**0.0543**	**n.c.**	**0.0565**	**0.0505**
C^{X-T+B}	**0.0543**	**0.0543**	**0.0543**	**0.1694**	**0.0460**	**n.c.**
$\Pi_T^{K^{X-T+B}}$ or $\rho_B^{K^{X-T+B}}$	**...**	**0.0000**	**...**	**0.1150**	**...**	**...**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
... Not applicable.

Table 34 shows that adding a progressive transfer to a fiscal system with a progressive tax (where progressivity is calculated with respect to households ranked by original income) could be unequalizing.

Table 34. Addition of a Progressive Transfer with an Unequalizing Effect to a Fiscal System with a Progressive Tax

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	10.00	7.00	17.00	1.00	9.00	16.00
2	11.00	9.00	20.00	1.00	10.00	19.00
3	12.00	9.00	21.00	1.90	10.10	19.10
4	13.00	9.00	22.00	2.80	10.20	19.20
Total	**46.00**	**34.00**	**80.00**	**6.70**	**39.30**	**73.30**
Average	**11.50**	**8.50**	**20.00**	**1.68**	**9.83**	**18.33**
Gini	**0.0543**	**n.c.**	**0.0500**	**n.c.**	**0.0235**	**0.0331**
C^{X-T+B}	**0.0543**	**0.0441**	**0.0500**	**0.2351**	**0.0235**	**n.c.**
$\Pi_T^{K^{X-T+B}}$ or $\rho_B^{K^{X-T+B}}$	**...**	**0.0102**	**...**	**0.1807**	**...**	**...**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
... Not applicable.

Although equation 35 is derived using the R-S index calculated with respect to the end income ranking of households, one can calculate a similar derivation using the R-S index calculated with respect to the original income ranking, as shown in the following equation:

$$M_B = \left(\frac{-b\Pi_T^{RS}+(1+b)\rho_B^{RS}}{1-g+b}\right) + [(C_{X-T+B}^{X} - G_{X-T+B}) - (C_{X-T}^{X} - G_{X-T})].$$

Because both terms in the brackets are non-positive, the bracket could be positive, zero, or negative. For the tax to be equalizing, the following condition should hold:

$$\left(\frac{-b\Pi_T^{RS} + (1+b)\rho_B^{RS}}{1-g+b}\right) + [(C_{X-T+B}^{X} - G_{X-T+B}) - (C_{X-T}^{X} - G_{X-T})] > 0$$

or

(38)

$$\left(\frac{-\frac{gb}{1-g}\Pi_T^{K}+b\rho_B^{K}}{1-g+b}\right) + \overbrace{\left[\underbrace{(C_{X-T+B}^{X} - G_{X-T+B})}_{Reranking\ after\ the\ transfer\ is\ added} - \underbrace{(C_{X-T}^{X} - G_{X-T})}_{Reranking\ before\ the\ transfer\ is\ added}\right]}^{Marginal\ effect\ of\ the\ transfer\ on\ reranking} > 0.$$

As table 35 shows, using Kakwani indexes calculated with respect to the original income ranking of households cannot give a definitive answer about the marginal effect of a transfer in any of the cases.

Table 35. Marginal Contribution of a Transfer with a Tax in Place

		Adding a Transfer that with respect to the original income ranking is		
		Regressive $\rho_B^K < 0$	Neutral $\rho_B^K = 0$	Progressive $\rho_B^K > 0$
To a system with a Tax that with respect to the original income ranking is	Regressive $\Pi_T^K < 0$	Equalizing if and only if condition 38 holds	Equalizing if and only if condition 38 holds	Equalizing if and only if condition 38 holds
	Neutral $\Pi_T^K = 0$	Equalizing if and only if condition 38 holds	Equalizing if and only if condition 38 holds	Equalizing if and only if condition 38 holds
	Progressive $\Pi_T^K > 0$	Equalizing if and only if condition 38 holds	Equalizing if and only if condition 38 holds	Equalizing if and only if condition 38 holds

In calculating progressivity, households' rank with respect to their original income is used.

Table 36 shows that adding a regressive transfer to a fiscal system with a regressive tax (where progressivity is calculated with respect to households ranked by original income) could be equalizing.

Table 36. Addition of a Regressive Transfer with an Equalizing Effect to a Fiscal System with a Regressive Tax

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.00	1.00	0.00	1.00	1.00
2	11.00	0.00	11.00	1.10	9.90	9.90
3	12.00	2.00	14.00	3.00	9.00	11.00
4	13.00	0.10	13.10	1.00	12.00	12.10
Total	**37.00**	**2.10**	**39.10**	**5.10**	**31.90**	**34.00**
Average	**9.25**	**0.53**	**9.78**	**1.28**	**7.98**	**8.50**
Gini	**0.2500**	**n.c.**	**0.2628**	**n.c.**	**0.2657**	**0.2529**
C^X	**0.2500**	**0.2738**	**0.2513**	**0.2402**	**0.2516**	**0.2529**
Π_T^K or ρ_B^K	**...**	**−0.0013**	**...**	**−0.0016**	**...**	**...**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
... Not applicable.

Table 37 shows that adding a neutral transfer to a fiscal system with a regressive tax (where progressivity is calculated with respect to households ranked by original income) could be equalizing.

Table 37. Addition of a Neutral Transfer with an Equalizing Effect to a Fiscal System with a Regressive Tax

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.00	1.00	0.00	1.00	1.00
2	11.00	0.00	11.00	1.10	9.90	9.90
3	12.00	2.00	14.00	3.00	9.00	11.00
4	13.00	0.00	13.00	1.00	12.00	12.00
Total	**37.00**	**2.00**	**39.00**	**5.10**	**31.90**	**33.90**
Average	**9.25**	**0.50**	**9.75**	**1.28**	**7.98**	**8.48**
Gini	**0.2500**	**n.c.**	**0.2628**	**n.c.**	**0.2657**	**0.2515**
C^X	**0.2500**	**0.2500**	**0.2500**	**0.2402**	**0.2516**	**0.2515**
Π_T^K or ρ_B^K	**...**	**0.0000**	**...**	**−0.0016**	**...**	**...**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
... Not applicable.

Table 38 shows that adding a neutral transfer to a fiscal system with a regressive tax (where progressivity is calculated with respect to households ranked by original income) could be unequalizing.

Table 38. Addition of a Neutral Transfer with an Unequalizing Effect to a Fiscal System with a Regressive Tax

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.00	1.00	0.00	1.00	1.00
2	11.00	0.00	11.00	1.10	9.90	9.90
3	12.00	5.00	17.00	3.00	9.00	14.00
4	13.00	0.00	13.00	1.00	12.00	12.00
Total	**37.00**	**5.00**	**42.00**	**5.10**	**31.90**	**36.90**
Average	**9.25**	**1.25**	**10.50**	**1.28**	**7.98**	**9.23**
Gini	**0.2500**	**n.c.**	**0.2976**	**n.c.**	**0.2657**	**0.2785**
C^X	**0.2500**	**0.2500**	**0.2500**	**0.2402**	**0.2516**	**0.2514**
Π_T^K or ρ_B^K	**...**	**0.0000**	**...**	**−0.0016**	**...**	**...**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
... Not applicable.

Table 39 shows that adding a progressive transfer to a fiscal system with a regressive tax (where progressivity is calculated with respect to households ranked by original income) could be unequalizing.

Table 39. Addition of a Progressive Transfer with an Unequalizing Effect to a Fiscal System with a Regressive Tax

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.10	1.10	0.00	1.00	1.10
2	11.00	0.00	11.00	1.10	9.90	9.90
3	12.00	5.00	17.00	3.00	9.00	14.00
4	13.00	0.00	13.00	1.00	12.00	12.00
Total	**37.00**	**5.10**	**42.10**	**5.10**	**31.90**	**37.00**
Average	**9.25**	**1.28**	**10.53**	**1.28**	**7.98**	**9.25**
Gini	**0.2500**	**n.c.**	**0.2951**	**n.c.**	**0.2657**	**0.2757**
C^X	**0.2500**	**0.2304**	**0.2476**	**0.2402**	**0.2516**	**0.2486**
Π_T^K or ρ_B^K	**...**	**0.0024**	**...**	**-0.0016**	**...**	**...**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
... Not applicable.

Table 40 shows that adding a regressive transfer to a fiscal system with a neutral tax (where progressivity is calculated with respect to households ranked by original income) could be equalizing.

Table 40. Addition of a Regressive Transfer with an Equalizing Effect to a Fiscal System with a Neutral Tax

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.00	1.00	0.00	1.00	1.00
2	11.00	0.00	11.00	1.00	10.00	10.00
3	12.00	3.00	15.00	3.00	9.00	12.00
4	13.00	0.10	13.10	1.00	12.00	12.10
Total	**37.00**	**3.10**	**40.10**	**5.00**	**32.00**	**35.10**
Average	**9.25**	**0.78**	**10.03**	**1.25**	**8.00**	**8.78**
Gini	**0.2500**	**n.c.**	**0.2749**	**n.c.**	**0.2656**	**0.2514**
C^X	**0.2500**	**0.2661**	**0.2512**	**0.2500**	**0.2500**	**0.2514**
Π_T^K or ρ_B^K	**…**	**−0.0012**	**…**	**0.0000**	**…**	**…**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
... Not applicable.

Table 41 shows that adding a neutral transfer to a fiscal system with a neutral tax (where progressivity is calculated with respect to households ranked by original income) could be equalizing.

Table 41. Addition of a Neutral Transfer with an Equalizing Effect to a Fiscal System with a Neutral Tax

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.00	1.00	0.00	1.00	1.00
2	11.00	0.00	11.00	1.00	10.00	10.00
3	12.00	2.00	14.00	3.00	9.00	11.00
4	13.00	0.00	13.00	1.00	12.00	12.00
Total	**37.00**	**2.00**	**39.00**	**5.00**	**32.00**	**34.00**
Average	**9.25**	**0.50**	**9.75**	**1.25**	**8.00**	**8.50**
Gini	**0.2500**	**n.c.**	**0.2628**	**n.c.**	**0.2656**	**0.2500**
C^X	**0.2500**	**0.2500**	**0.2500**	**0.2500**	**0.2500**	**0.2500**
Π_T^K or ρ_B^K	**…**	**0.0000**	**…**	**0.0000**	**…**	**…**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
... Not applicable.

Table 42 shows that adding a neutral transfer to a fiscal system with a neutral tax (where progressivity is calculated with respect to households ranked by original income) could be unequalizing.

Table 42. Addition of a Neutral Transfer with an Unequalizing Effect to a Fiscal System with a Neutral Tax

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.00	1.00	0.00	1.00	1.00
2	11.00	0.00	11.00	1.00	10.00	10.00
3	12.00	5.00	17.00	3.00	9.00	14.00
4	13.00	0.00	13.00	1.00	12.00	12.00
Total	**37.00**	**5.00**	**42.00**	**5.00**	**32.00**	**37.00**
Average	**9.25**	**1.25**	**10.50**	**1.25**	**8.00**	**9.25**
Gini	**0.2500**	**n.c.**	**0.2976**	**n.c.**	**0.2656**	**0.2770**
C^X	**0.2500**	**0.2500**	**0.2500**	**0.2500**	**0.2500**	**0.2500**
Π_T^K or ρ_B^K	**…**	**0.0000**	**…**	**0.0000**	**…**	**…**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
... Not applicable.

Table 43 shows that adding a progressive transfer to a fiscal system with a neutral tax (where progressivity is calculated with respect to households ranked by original income) could be unequalizing.

Table 43. Addition of a Progressive Transfer with an Unequalizing Effect to a Fiscal System with a Neutral Tax

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.10	1.10	0.00	1.00	1.10
2	11.00	0.00	11.00	1.00	10.00	10.00
3	12.00	5.00	17.00	3.00	9.00	14.00
4	13.00	0.00	13.00	1.00	12.00	12.00
Total	**37.00**	**5.10**	**42.10**	**5.00**	**32.00**	**37.10**
Average	**9.25**	**1.28**	**10.53**	**1.25**	**8.00**	**9.28**
Gini	**0.2500**	**n.c.**	**0.2951**	**n.c.**	**0.2656**	**0.2743**
C^X	**0.2500**	**0.2304**	**0.2476**	**0.2500**	**0.2500**	**0.2473**
Π_T^K or ρ_B^K	**…**	**0.0024**	**…**	**0.0000**	**…**	**…**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
... Not applicable.

Table 44 shows that adding a regressive transfer to a fiscal system with a progressive tax (where progressivity is calculated with respect to households ranked by original income) could be equalizing.

Table 44. Addition of a Regressive Transfer with an Equalizing Effect to a Fiscal System with a Progressive Tax

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.00	1.00	0.00	1.00	1.00
2	11.00	0.00	11.00	1.00	10.00	10.00
3	12.00	2.00	14.00	3.00	9.00	11.00
4	13.00	0.10	13.10	1.10	11.90	12.00
Total	**37.00**	**2.10**	**39.10**	**5.10**	**31.90**	**34.00**
Average	**9.25**	**0.53**	**9.78**	**1.28**	**7.98**	**8.50**
Gini	**0.2500**	**n.c.**	**0.2628**	**n.c.**	**0.2641**	**0.2500**
C^X	**0.2500**	**0.2738**	**0.2513**	**0.2598**	**0.2484**	**0.2500**
Π_T^K or ρ_B^K	**...**	**−0.0013**	**...**	**0.0016**	**...**	**...**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
... Not applicable.

Table 45 shows that adding a neutral transfer to a fiscal system with a progressive tax (where progressivity is calculated with respect to households ranked by original income) could be unequalizing.

Table 45. Addition of a Neutral Transfer with an Unequalizing Effect to a Fiscal System with a Progressive Tax

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.00	1.00	0.00	1.00	1.00
2	11.00	0.00	11.00	1.00	10.00	10.00
3	12.00	5.00	17.00	3.00	9.00	14.00
4	13.00	0.00	13.00	1.10	11.90	11.90
Total	**37.00**	**5.00**	**42.00**	**5.10**	**31.90**	**36.90**
Average	**9.25**	**1.25**	**10.50**	**1.28**	**7.98**	**9.23**
Gini	**0.2500**	**n.c.**	**0.2976**	**n.c.**	**0.2641**	**0.2771**
C^X	**0.2500**	**0.2500**	**0.2500**	**0.2598**	**0.2484**	**0.2486**
Π_T^K or ρ_B^K	**...**	**0.0000**	**...**	**0.0016**	**...**	**...**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
... Not applicable.

Table 46 shows that adding a progressive transfer to a fiscal system with a progressive tax (where progressivity is calculated with respect to households ranked by original income) could be unequalizing.

Table 46. Addition of a Progressive Transfer with an Unequalizing Effect to a Fiscal System with a Progressive Tax

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.10	1.10	0.00	1.00	1.10
2	11.00	0.00	11.00	1.00	10.00	10.00
3	12.00	5.00	17.00	3.00	9.00	14.00
4	13.00	0.00	13.00	1.10	11.90	11.90
Total	**37.00**	**5.10**	**42.10**	**5.10**	**31.90**	**37.00**
Average	**9.25**	**1.28**	**10.53**	**1.28**	**7.98**	**9.25**
Gini	**0.2500**	**n.c.**	**0.2951**	**n.c.**	**0.2641**	**0.2743**
C^X	**0.2500**	**0.2304**	**0.2476**	**0.2598**	**0.2484**	**0.2459**
Π_T^K **or** ρ_B^K	**...**	**0.0024**	**...**	**0.0016**	**...**	**...**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
... Not applicable.

4.3. The Case of Adding a Transfer to a System with Multiple Taxes and Transfers in Place

Recall from equation 14 that

$$M_{B_1} = \left\{(G_X - C_X^Z) + \left(\frac{\sum_{i=1}^{n}(1-g_i)\Pi_{T_i}^{RS^Z} + \sum_{j=1}^{m}(1+b_j)\rho_{B_j}^{RS^Z}}{1-\sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j}\right)\right\} - \left\{(G_X - C_X^{Z\backslash B_1}) + \left(\frac{\sum_{i=1}^{n}(1-g_i)\Pi_{T_i}^{RS^{Z\backslash B_1}} + \sum_{j=2}^{m}(1+b_j)\rho_{B_j}^{RS^{Z\backslash B_1}}}{1-\sum_{i=1}^{n} g_i + \sum_{j=2}^{m} b_j}\right)\right\}.$$

For B_1 to be equalizing, this equation has to be positive, that is,

(39)

$$\left\{(\boldsymbol{G_X} - \boldsymbol{C_X^Z}) + \left(\frac{\sum_{i=1}^{n}(1-g_i)\Pi_{T_i}^{RS^Z} + \sum_{j=1}^{m}(1+b_j)\rho_{B_j}^{RS^Z}}{1-\sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j}\right)\right\} - \left\{(\boldsymbol{G_X} - \boldsymbol{C_X^{Z\backslash B_1}}) + \left(\frac{\sum_{i=1}^{n}(1-g_i)\Pi_{T_i}^{RS^{Z\backslash B_1}} + \sum_{j=2}^{m}(1+b_j)\rho_{B_j}^{RS^{Z\backslash B_1}}}{1-\sum_{i=1}^{n} g_i + \sum_{j=2}^{m} b_j}\right)\right\} > 0.$$

If adding this specific transfer does not change the end income ranking of individuals (that is, if end income rankings are the same before and after adding the tax), then ranking with respect to Z and $Z\backslash B_1$ is the same, which simplifies the whole equation to

$$\left(1-\sum_{i=1}^{n} g_i+\sum_{j=2}^{m} b_j\right)(1+b_1)\rho_{B_1}^{RS^Z} > b_1\left(\sum_{i=1}^{n}(1-g_i)\Pi_{T_i}^{RS^{Z\backslash B_1}}+\sum_{j=2}^{m}\left(1+b_j\right)\rho_{B_j}^{RS^{Z\backslash B_1}}\right)$$

which is equal to

$$\rho_{B_1}^{RS^Z} > \frac{b_1}{1+b_1}\left(\frac{\sum_{i=1}^{n}(1-g_i)\Pi_{T_i}^{RS^{Z\backslash B_1}}+\sum_{j=2}^{m}\left(1+b_j\right)\rho_{B_j}^{RS^{Z\backslash B_1}}}{1-\sum_{i=1}^{n} g_i+\sum_{j=2}^{m} b_j}\right)$$

or

$$\rho_{B_1}^{RS^Z} > \frac{b_1}{1+b_1}\left(C_X^{Z\backslash B_1}-G_{Z\backslash B_1}\right)$$

or

(40)

$$\boldsymbol{\rho_{B_1}^{K^Z} > \left(C_X^{Z\backslash B_1}-G_{Z\backslash B\_1}\right)}.$$

As mentioned in section 3.3, the term on the right-hand side is

$VE_{X,Z\backslash B_1}^{Z\backslash B_1} = C_X^{Z\backslash B_1} - G_{Z\backslash B_1}$.

Thus,

(41) $$\boldsymbol{\rho_{B_1}^{K^Z} > VE_{X,Z\backslash B_1}^{Z\backslash B_1}}$$

Therefore, we can use table 47 to determine the marginal effect of adding a transfer to a system with multiple taxes and transfers when the end income ranking of households does not change because of this additional transfer.

Table 47. Marginal Contribution of a Transfer with Multiple Taxes and Transfers in Place

		To a system with multiple taxes and transfers where its vertical equity (with respect to the final income ranking) is		
		Negative $VE^{Z\backslash B_1}_{X,Z\backslash B_1} < 0$	Zero $VE^{Z\backslash B_1}_{X,Z\backslash B_1} = 0$	Positive $VE^{Z\backslash B_1}_{X,Z\backslash B_1} > 0$
Adding a Transfer that with respect to the final incomes ranking (Z) is	Regressive $\rho^{K^Z}_B < 0$	Equalizing if and only if condition 41 holds	Always Unequalizing	Always Unequalizing
	Neutral $\rho^{K^Z}_B = 0$	Always Equalizing	No Change in Inequality	Always Unequalizing
	Progressive $\rho^{K^Z}_B > 0$	Always Equalizing	Always Equalizing	Equalizing if and only if condition 41 holds

$Z = X - \sum_{i=1}^{n} T_i + \sum_{j=1}^{m} B_j$ and $Z\backslash B_1 = X - \sum_{i=1}^{n} T_i + \sum_{j=2}^{m} B_j$. Adding the new transfer does not change the end income ranking of individuals.

Crucially, for the preceding results to hold, the transfer that we are interested in should not have any effect on the end income ranking of households. If that is not the case then equation 39 cannot be simplified much further and the effect of adding such a transfer cannot be determined using a simple rule of thumb from table 47.

Alternatively, one can use the progressivity with respect to the original income in the analysis. For this purpose, we need to use equation 18:

$$\boldsymbol{M_{B_1}} = \left[\left(\frac{\left[(\mathbf{1}+\boldsymbol{b_1})\boldsymbol{\rho^{RS}_{B_1}}\right] - \left[(\boldsymbol{b_1}) \overbrace{\left(\boldsymbol{G_X} - \boldsymbol{C^X_{Z\backslash B_1}}\right)}^{VE\ of\ the\ system\ without\ B_1}\right]}{\left(\mathbf{1} - \sum_{i=1}^{n} \boldsymbol{g_i} + \sum_{j=1}^{m} \boldsymbol{b_j}\right)}\right)\right] + \underbrace{\left[\left(\boldsymbol{C^X_Z} - \boldsymbol{G_Z}\right) - \left(\boldsymbol{C^X_{Z\backslash B_1}} - \boldsymbol{G_{Z\backslash B_1}}\right)\right]}_{Contribution\ of\ B_1 to\ reranking}.$$

For a transfer to be equalizing when it is added to a system of taxes and transfers, the following condition should hold:

(42)

$$\boldsymbol{M_{B_1}} = \left[\left(\frac{\left[(\mathbf{1}+\boldsymbol{b_1})\boldsymbol{\rho^{RS}_{B_1}}\right] - \left[(\boldsymbol{b_1}) \overbrace{\left(\boldsymbol{G_X} - \boldsymbol{C^X_{Z\backslash B_1}}\right)}^{VE\ of\ the\ system\ without\ B_1}\right]}{\left(\mathbf{1} - \sum_{i=1}^{n} \boldsymbol{g_i} + \sum_{j=1}^{m} \boldsymbol{b_j}\right)}\right)\right] + \underbrace{\left[\left(\boldsymbol{C^X_Z} - \boldsymbol{G_Z}\right) - \left(\boldsymbol{C^X_{Z\backslash B_1}} - \boldsymbol{G_{Z\backslash B_1}}\right)\right]}_{Contribution\ of\ B_1 to\ reranking} > 0$$

or

(43)

$$\boldsymbol{M_{B_1}} = \left[\left(\frac{\left[b_1\rho_{B_1}^K\right] - \left[(b_1)\overbrace{\left(G_X - C_{Z\backslash B_1}^X\right)}^{VE\ of\ the\ system\ without\ B_1}\right]}{\left(1 - \sum_{i=1}^n g_i + \sum_{j=1}^m b_j\right)}\right)\right] + \underbrace{\left[\left(\boldsymbol{C_Z^X} - \boldsymbol{G_Z}\right) - \left(\boldsymbol{C_{Z\backslash B_1}^X} - \boldsymbol{G_{Z\backslash B_1}}\right)\right]}_{Contribution\ of\ B_1 to\ reranking} > 0\ .$$

5. Is the Total System More Equal?: Adding a Tax and a Transfer

After examining the marginal contribution of taxes and transfers in the previous two sections, this section examines the total redistributive effect of all taxes and transfers. For simplicity, I bundle all of the taxes together and all of the transfers together and treat them as if there were only one tax and one transfer in the system. Recall that the change in the Gini is equal to

$$G_X - G_{X-T+B} = (G_X - C_X^{X-T+B}) + \left(\frac{(1-g)\Pi_T^{RS^{X-T+B}} + (1+b)\rho_B^{RS^{X-T+B}}}{1-g+b}\right).$$

Then, for the whole system to be equalizing, we would need the following condition to hold:

(44)

$$(G_X - C_X^{X-T+B}) + \left(\frac{(1-g)\Pi_T^{RS^{X-T+B}} + (1+b)\rho_B^{RS^{X-T+B}}}{1-g+b}\right) > 0$$

or

(45)

$$(G_X - C_X^{X-T+B}) + \left(\frac{g\Pi_T^{K^{X-T+B}} + b\rho_B^{K^{X-T+B}}}{1-g+b}\right) > 0.$$

Note that the first term is non-negative. Therefore, we have the following cases. Table 48 shows the effect of the total system in the case of one tax and one transfer and when progressivity is calculated with respect to the end income ranking of households.

Table 48. Effect of the Total System with One Tax and One Transfer

		If the Transfer with respect to the end income ranking is		
		Regressive $\rho_B^{K^{X-T+B}} < 0$	Neutral $\rho_B^{K^{X-T+B}} = 0$	Progressive $\rho_B^{K^{X-T+B}} > 0$
If the Tax with respect to the end income ranking is	Regressive $\Pi_T^{K^{X-T+B}} < 0$	Equalizing if and only if 45 holds	Equalizing if and only if equation 45 holds	Equalizing if and only if equation 45 holds
	Neutral $\Pi_T^{K^{X-T+B}} = 0$	Equalizing if and only if equation 45 holds	Equalizing if and only if equation 45 holds	Always equalizing
	Progressive $\Pi_T^{K^{X-T+B}} > 0$	Equalizing if and only if equation 45 holds	Always equalizing	Always equalizing

In calculating progressivity, households' rank with respect to their original income is used.

The following examples display the counterintuitive cases.

Table 49 shows that adding a regressive tax and a regressive transfer (where progressivity is calculated with respect to households ranked by original income) to a fiscal system could be equalizing.

Table 49. Addition of a Regressive Tax and a Regressive Transfer with an Equalizing Effect to a Fiscal System

Individual	**Original Income (OI)**	**Benefit (B)**	**OI + B**	**Tax (T)**	**OI – T**	**End Income (EI)**
1	1.00	12.10	13.10	1.00	0.00	12.10
2	11.00	0.00	11.00	0.00	11.00	11.00
3	12.00	0.00	12.00	10.00	2.00	2.00
4	13.00	0.00	13.00	1.10	11.90	11.90
Total	**37.00**	**12.10**	**49.10**	**12.10**	**24.90**	**37.00**
Average	**9.25**	**3.03**	**12.28**	**3.03**	**6.23**	**9.25**
Gini	**0.2500**	**n.c.**	**0.0372**	**n.c.**	**0.4488**	**0.2108**
$\boldsymbol{C^{X-T+B}}$	**−0.2095**	**0.7500**	**0.0270**	**-0.5351**	**-0.0512**	**n.c.**
$\boldsymbol{\Pi_T^{K^{X-T+B}}}$ **or** $\boldsymbol{\rho_B^{K^{X-T+B}}}$	**…**	**−0.9595**	**…**	**−0.3257**	**…**	**…**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
... Not applicable.

Table 50 shows that adding a regressive tax and a neutral transfer (where progressivity is calculated with respect to households ranked by original income) to a fiscal system could be equalizing.

Table 50. Addition of a Regressive Tax and a Neutral Transfer with an Equalizing Effect to a Fiscal System

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.10	1.10	0.10	0.90	1.00
2	11.00	1.10	12.10	1.10	9.90	11.00
3	12.00	1.20	13.20	1.20	10.80	12.00
4	13.00	1.30	14.30	3.40	9.60	10.90
Total	**37.00**	**3.70**	**40.70**	**5.80**	**31.20**	**34.90**
Average	**9.25**	**0.93**	**10.18**	**1.45**	**7.80**	**8.73**
Gini	**0.2500**	**n.c.**	**0.2500**	**n.c.**	**0.2404**	**0.2371**
C^{X-T+B}	**0.2095**	**0.2095**	**0.2095**	**0.0431**	**0.2404**	**n.c.**
$\Pi_T^{K^{X-T+B}}$ **or** $\rho_B^{K^{X-T+B}}$	**...**	**0.0000**	**...**	**−0.1664**	**...**	**...**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
... Not applicable.

Table 51 shows that adding a neutral tax and a regressive transfer (where progressivity is calculated with respect to households ranked by original income) to a fiscal system could be equalizing.

Table 51. Addition of a Neutral Tax and a Regressive Transfer with an Equalizing Effect to a Fiscal System

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	10.10	11.10	0.40	0.60	10.70
2	11.00	0.00	11.00	4.40	6.60	6.60
3	12.00	0.00	12.00	4.80	7.20	7.20
4	13.00	0.00	13.00	5.20	7.80	7.80
Total	**37.00**	**10.10**	**47.10**	**14.80**	**22.20**	**32.30**
Average	**9.25**	**2.53**	**11.78**	**3.70**	**5.55**	**8.08**
Gini	**0.2500**	**n.c.**	**0.0366**	**n.c.**	**0.2500**	**0.0998**
C^{X-T+B}	**−0.1959**	**0.7500**	**0.0069**	**−0.1959**	**−0.1959**	**n.c.**
$\Pi_T^{K^{X-T+B}}$ **or** $\rho_B^{K^{X-T+B}}$	**...**	**−0.9459**	**...**	**0.0000**	**...**	**...**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
... Not applicable.

Table 52 shows that adding a neutral tax and a neutral transfer (where progressivity is calculated with respect to households ranked by original income) to a fiscal system could be equalizing.

Table 52. Addition of a Neutral Tax and a Neutral Transfer with an Equalizing Effect to a Fiscal System

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.0000	0.2000	1.2000	0.0000	1.0000	1.2000
2	11.0000	2.2000	13.2000	1.0148	9.9852	12.1850
3	12.0000	2.4000	14.4000	3.0000	9.0000	11.4000
4	13.0000	2.6000	15.6000	2.8154	10.1846	12.7850
Total	**37.0000**	**7.4000**	**44.4000**	**6.8302**	**30.1698**	**37.5698**
Average	**9.2500**	**1.8500**	**11.1000**	**1.7076**	**7.5425**	**9.3925**
Gini	**0.2500**	**n.c.**	**0.2500**	**n.c.**	**0.2365**	**0.2365**
C^{X-T+B}	**0.2365**	**0.2365**	**0.2365**	**0.2365**	**0.2365**	**n.c.**
$\Pi_T^{K^{X-T+B}}$ or $\rho_B^{K^{X-T+B}}$	**...**	**0.0000**	**...**	**0.0000**	**...**	**...**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
... Not applicable.

Alternatively, we can use the formula based on the Kakwani index calculated with respect to the original income ranking of households:

$$G_X - G_{X-T+B} = (G_X - C_{X-T+B}^X) + (C_{X-T+B}^X - G_{X-T+B})$$

which can be written as

$$G_X - G_{X-T+B} = \left(\frac{(1-g)\Pi_T^{RS}+(1+b)\rho_B^{RS}}{1-g+b}\right) + (C_{X-T+B}^X - G_{X-T+B}).$$

For the total system to be equalizing, we need to have

(46)

$$\left(\frac{(1-g)\Pi_T^{RS}+(1+b)\rho_B^{RS}}{1-g+b}\right) + (C_{X-T+B}^X - G_{X-T+B}) > 0$$

or

(47) $$\left(\frac{g\Pi_T^K+b\rho_B^K}{1-g+b}\right) + (C_{X-T+B}^X - G_{X-T+B}) > 0.$$

Note that the latter term is always non-positive. Therefore, we have the following cases.

Table 53 shows the effect of the total system in the case of one tax and one transfer and when progressivity is calculated with respect to the original income ranking of households.

Table 53. The Effect of the Total System with One Tax and One Transfer

		If the Transfer with respect to the original income ranking is		
		Regressive $\rho_B^K < 0$	Neutral $\rho_B^K = 0$	Progressive $\rho_B^K > 0$
If the Tax with respect to the original income ranking is	Regressive $\Pi_T^K < 0$	Always unequalizing	Always unequalizing	Equalizing if and only if equation 47 holds
	Neutral $\Pi_T^K = 0$	Always unequalizing	Never equalizing	Equalizing if and only if equation 47 holds
	Progressive $\Pi_T^K > 0$	Equalizing if and only if equation 47 holds	Equalizing if and only if equation 47 holds	Equalizing if and only if equation 47 holds

In calculating progressivity, households' rank with respect to their original income is used.

The relatively counterintuitive cases in table 53 are presented in the following examples.

Table 54 shows that adding a neutral tax and a neutral transfer (where progressivity is calculated with respect to households ranked by original income) to a fiscal system could be unequalizing.

Table 54. Addition of a Neutral Tax and a Neutral Transfer with an Unequalizing Effect to a Fiscal System

Individual	**Original Income (OI)**	**Benefit (B)**	**OI + B**	**Tax (T)**	**OI – T**	**End Income (EI)**
1	1.00	0.00	1.00	0.00	1.00	1.00
2	11.00	0.00	11.00	1.00	10.00	10.00
3	12.00	5.00	17.00	3.00	9.00	14.00
4	13.00	0.00	13.00	1.00	12.00	12.00
Total	**37.00**	**5.00**	**42.00**	**5.00**	**32.00**	**37.00**
Average	**9.25**	**1.25**	**10.50**	**1.25**	**8.00**	**9.25**
Gini	**0.2500**	**n.c.**	**0.2976**	**n.c.**	**0.2656**	**0.2770**
C^X	**0.2500**	**0.2500**	**0.2500**	**0.2500**	**0.2500**	**0.2500**
Π_T^K or ρ_B^K	**...**	**0.0000**	**...**	**0.0000**	**...**	**...**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
... Not applicable.

Table 55 shows that adding a neutral tax and a progressive transfer (where progressivity is calculated with respect to households ranked by original income) to a fiscal system could be unequalizing.

Table 55. Addition of a Neutral Tax and a Progressive Transfer with an Unequalizing Effect to a Fiscal System

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.10	1.10	0.00	1.00	1.10
2	11.00	0.00	11.00	1.00	10.00	10.00
3	12.00	5.00	17.00	3.00	9.00	14.00
4	13.00	0.00	13.00	1.00	12.00	12.00
Total	**37.00**	**5.10**	**42.10**	**5.00**	**32.00**	**37.10**
Average	**9.25**	**1.28**	**10.53**	**1.25**	**8.00**	**9.28**
Gini	**0.2500**	**n.c.**	**0.2951**	**n.c.**	**0.2656**	**0.2743**
C^X	**0.2500**	**0.2304**	**0.2476**	**0.2500**	**0.2500**	**0.2473**
Π_T^K or ρ_B^K	**...**	**0.0024**	**...**	**0.0000**	**...**	**...**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
... Not applicable.

Table 56 shows that adding a progressive tax and a neutral transfer (where progressivity is calculated with respect to households ranked by original income) to a fiscal system could be unequalizing.

Table 56. Addition of a Progressive Tax and a Neutral Transfer with an Unequalizing Effect to a Fiscal System

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.00	1.00	0.00	1.00	1.00
2	11.00	0.00	11.00	1.00	10.00	10.00
3	12.00	5.00	17.00	3.00	9.00	14.00
4	13.00	0.00	13.00	1.10	11.90	11.90
Total	**37.00**	**5.00**	**42.00**	**5.10**	**31.90**	**36.90**
Average	**9.25**	**1.25**	**10.50**	**1.28**	**7.98**	**9.23**
Gini	**0.2500**	**n.c.**	**0.2976**	**n.c.**	**0.2641**	**0.2771**
C^X	**0.2500**	**0.2500**	**0.2500**	**0.2598**	**0.2484**	**0.2486**
Π_T^K or ρ_B^K	**...**	**0.0000**	**...**	**0.0016**	**...**	**...**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
... Not applicable.

Table 57 shows that adding a progressive tax and a progressive transfer (where progressivity is calculated with respect to households ranked by original income) to a fiscal system could be unequalizing.

Table 57. Addition of a Progressive Tax and a Progressive Transfer with an Unequalizing Effect to a Fiscal System

Individual	Original Income (OI)	Benefit (B)	OI + B	Tax (T)	OI – T	End Income (EI)
1	1.00	0.10	1.10	0.00	1.00	1.10
2	11.00	0.00	11.00	1.00	10.00	10.00
3	12.00	5.00	17.00	3.00	9.00	14.00
4	13.00	0.00	13.00	1.10	11.90	11.90
Total	**37.00**	**5.10**	**42.10**	**5.10**	**31.90**	**37.00**
Average	**9.25**	**1.28**	**10.53**	**1.28**	**7.98**	**9.25**
Gini	**0.2500**	**n.c.**	**0.2951**	**n.c.**	**0.2641**	**0.2743**
C^X	**0.2500**	**0.2304**	**0.2476**	**0.2598**	**0.2484**	**0.2459**
Π_T^K or ρ_B^K	**...**	**0.0024**	**...**	**0.0016**	**...**	**...**

In calculating progressivity, households' rank with respect to their original income is used.
n.c. Not calculated.
... Not applicable.

6. The Effect of a Marginal Change in One Tax or Transfer on the Equalizing (Unequalizing) Effect of a Whole System

This section focuses on the derivatives of the marginal contribution of a tax or transfer (that is, M_{T_1} or M_{B_1}), with respect to its progressivity or relative size, to determine whether such a marginal change would increase the equalizing effect of the whole system. What differentiates this section from the previous paper[11] (the case of no-reranking) is that the progressivity is calculated with respect to both the end income ranking and to the original income ranking of households. In this section, therefore, I will discuss three derivatives (with respect to the relative size and two types of Kakwani indexes).

Before calculating the derivatives, I need to point out an important simplifying assumption. The derivatives represent a very minor change in a tax or transfer and therefore it is safe to assume that the end income ranking of households would not change. This is not the case, of course, if we deviate from the case of a very "marginal" change in a tax or transfer.

It should also be noted that, conceptually, the derivatives of marginal contribution with respect to either relative size or Kakwani indexes are equivalent to the derivatives of the redistributive

[11] Enami and others (2017).

effect or Gini of the end income with respect to these two variables, which should be easily seen in the following equation.[12]

$$M_{T_1} = G_{Z\backslash T_1} - G_Z = \overbrace{(G_X - G_Z)}^{RE} - \left(G_X - G_{Z\backslash T_1}\right)$$

Note that the Gini of the final income is the only term on the right-hand side that has T_1 in it, that is, G_Z and the rest of terms are constants in any derivative with respect to the relative size or Kakwani index of T_1 (and they would drop out). Also note that while the sign of the derivatives of G_Z is different from RE and $\mathrm{M}_{\mathrm{T}_1}$, they are of the same size and equivalent interpretation. To provide a more intuitive explanation, note how the following three statements in the example below are equivalent.

Example: Due to a marginal change in a tax's relative size (or its progressivity),

---the end Gini decreased by 0.2.
---the redistributive effect of the total system increased by 0.2.
---the marginal contribution of that tax (to reducing inequality) increased by 0.2.

6.1. The Case of a Marginal Change in a Tax

This section focuses on the derivatives of the marginal contribution of a tax with respect to its relative size (g), Kakwani index calculated with respect to the original income ranking of households ($\Pi_{\boldsymbol{T}}^{\boldsymbol{K}}$), and Kakwani index calculated with respect to the end income ranking of households ($\Pi_{\boldsymbol{T}}^{\boldsymbol{K}^{\boldsymbol{Z}}}$).

To calculate the derivative of M_{T_1} with respect to g_1, we have two formulas to work with. Using equation 13,

$$\boldsymbol{M_{T_1}} = \left\{\left(\boldsymbol{G_X} - \boldsymbol{C_X^Z}\right) + \left(\frac{\sum_{i=1}^{n}(1-g_i)\Pi_{T_i}^{RS^Z} + \sum_{j=1}^{m}(1+b_j)\rho_{B_j}^{RS^Z}}{1-\sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j}\right)\right\} - \left\{\left(\boldsymbol{G_X} - \boldsymbol{C_X^{Z\backslash T_1}}\right) + \left(\frac{\sum_{i=2}^{n}(1-g_i)\Pi_{T_i}^{RS^{Z\backslash T_1}} + \sum_{j=1}^{m}(1+b_j)\rho_{B_j}^{RS^{Z\backslash T_1}}}{1-\sum_{i=2}^{n} g_i + \sum_{j=1}^{m} b_j}\right)\right\}$$

or

$$\boldsymbol{M_{T_1}} = \left\{\left(\boldsymbol{G_X} - \boldsymbol{C_X^Z}\right) + \left(\frac{\sum_{i=1}^{n} g_i\Pi_{T_i}^{K^Z} + \sum_{j=1}^{m} b_j\rho_{B_j}^{K^Z}}{1-\sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j}\right)\right\} - \left\{\left(\boldsymbol{G_X} - \boldsymbol{C_X^{Z\backslash T_1}}\right) + \left(\frac{\sum_{i=2}^{n} g_i\Pi_{T_i}^{K^{Z\backslash T_1}} + \sum_{j=1}^{m} b_j\rho_{B_j}^{K^{Z\backslash T_1}}}{1-\sum_{i=2}^{n} g_i + \sum_{j=1}^{m} b_j}\right)\right\}.$$

[12] Recall from the notation section that $Z = X - \sum_{i=1}^{n} T_i + \sum_{j=1}^{m} B_j$ and $Z\backslash T_1 = X - \sum_{i=2}^{n} T_i + \sum_{j=1}^{m} B_j$.

Therefore,

$$\frac{\partial M_{T_1}}{\partial g_1} = \frac{\partial(-C_X^Z)}{\partial g_1} + \frac{\left(1-\sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j\right)\left(\Pi_{T_1}^{K^Z} + \frac{\partial \Pi_{T_1}^{K^Z}}{\partial g_1} g_1 + \sum_{i=2}^{n} g_i \frac{\partial \Pi_{T_i}^{K^Z}}{\partial g_1} + \sum_{j=1}^{m} b_j \frac{\partial \rho_{B_j}^{K^Z}}{\partial g_1}\right) + \left(\sum_{i=1}^{n} g_i \Pi_{T_i}^{K^Z} + \sum_{j=1}^{m} b_j \rho_{B_j}^{K^Z}\right)}{\left(1-\sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j\right)^2}$$

or

$$\frac{\partial M_{T_1}}{\partial g_1} = \frac{\partial(-C_X^Z)}{\partial g_1} + \frac{\left(\Pi_{T_1}^{K^Z} + \frac{\partial \Pi_{T_1}^{K^Z}}{\partial g_1} g_1 + \sum_{i=2}^{n} g_i \frac{\partial \Pi_{T_i}^{K^Z}}{\partial g_1} + \sum_{j=1}^{m} b_j \frac{\partial \rho_{B_j}^{K^Z}}{\partial g_1}\right) + (C_X^Z - G_Z)}{1-\sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j}$$

Note that if a new reranking were to occur due to the marginal change in g_1, then all terms ordered by Z would change, thus making it impossible to derive any general conclusion. However, our assumption about no further reranking (with respect to the end income ranking of households) would simplify the above derivative to the following equation:

$$\frac{\partial M_{T_1}}{\partial g_1} = \frac{\Pi_{T_1}^{K^Z} + (C_X^Z - G_Z)}{1-\sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j} = \frac{C_{T_1}^Z - G_Z}{1-\sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j}.$$

The sign of this derivative is ambiguous. A closer look at the numerator reveals that it follows the same idea of the traditional Kakwani index. In other words, if the concentration curve of a tax (with respect to the end income concept) happens to be below the Gini of the end income, then a marginal increase in the size of that tax would increase the value of the marginal contribution of that tax (to reducing inequality). The other obvious case is when the concentration coefficient of a tax (with respect to the end income ranking of households) is negative, it makes the derivative unambiguously negative. This happens, for example, if the poorer a household is (with respect to the end income ranking of households), the more tax dollars it pays.

An equivalent formula can be derived from equation 16. From this equation, we have

$$M_{T_1} = \left[\left(\frac{g_1\left(\Pi_{T_1}^{K} - G_X - C_{Z\backslash T_1}^X\right)}{\left(1-\sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j\right)}\right)\right] + \left[(C_Z^X - G_Z) - \left(C_{Z\backslash T_1}^X - G_{Z\backslash T_1}\right)\right].$$

The derivative therefore is equal to

$$\frac{\partial \boldsymbol{M}_{\boldsymbol{T}_1}}{\partial \boldsymbol{g}_1} = \frac{\left(\Pi_{\boldsymbol{T}_1}^{\boldsymbol{K}} - \boldsymbol{G}_{\boldsymbol{X}} - \boldsymbol{C}_{\boldsymbol{Z}\setminus\boldsymbol{T}_1}^{\boldsymbol{X}}\right)\left(\mathbf{1} - \sum_{i=1}^{n} \boldsymbol{g}_i + \sum_{j=1}^{m} \boldsymbol{b}_j\right) + \boldsymbol{g}_1\left(\Pi_{\boldsymbol{T}_1}^{\boldsymbol{K}} - \boldsymbol{G}_{\boldsymbol{X}} - \boldsymbol{C}_{\boldsymbol{Z}\setminus\boldsymbol{T}_1}^{\boldsymbol{X}}\right)}{\left(\mathbf{1} - \sum_{i=1}^{n} \boldsymbol{g}_i + \sum_{j=1}^{m} \boldsymbol{b}_j\right)^2} + \frac{\partial\left(\boldsymbol{C}_{\boldsymbol{Z}}^{\boldsymbol{X}} - \boldsymbol{G}_{\boldsymbol{Z}}\right)}{\partial \boldsymbol{g}_1}$$

or

$$\frac{\partial \boldsymbol{M}_{\boldsymbol{T}_1}}{\partial \boldsymbol{g}_1} = \frac{\left(\Pi_{\boldsymbol{T}_1}^{\boldsymbol{K}} - \boldsymbol{G}_{\boldsymbol{X}} - \boldsymbol{C}_{\boldsymbol{Z}\setminus\boldsymbol{T}_1}^{\boldsymbol{X}}\right)\left(\mathbf{1} - \sum_{i=2}^{n} \boldsymbol{g}_i + \sum_{j=1}^{m} \boldsymbol{b}_j\right)}{\left(\mathbf{1} - \sum_{i=1}^{n} \boldsymbol{g}_i + \sum_{j=1}^{m} \boldsymbol{b}_j\right)^2} + \frac{\partial\left(\boldsymbol{C}_{\boldsymbol{Z}}^{\boldsymbol{X}} - \boldsymbol{G}_{\boldsymbol{Z}}\right)}{\partial \boldsymbol{g}_1}$$

Unlike the previous derivative, however, there is no reasonable simplifying assumption to take care of the last term,

$$\frac{\boldsymbol{\partial}(C_Z^X - G_Z)}{\boldsymbol{\partial}\mathbf{g}_1}.$$

In order to calculate the derivative with respect to the Kakwani index when this index is calculated with respect to the original income ranking of households, one needs to use equation 16 and the transformation of the R-S index to the Kakwani index as mentioned previously.

$$M_{T_1} = \left[\left(\frac{g_1\left(\Pi_{T_1}^K - G_X - C_{Z\setminus T_1}^X\right)}{\left(1 - \sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j\right)}\right)\right] + \left[(C_Z^X - G_Z) - \left(C_{Z\setminus T_1}^X - G_{Z\setminus T_1}\right)\right]$$

Therefore,

$$\frac{\partial M_{T_1}}{\partial \Pi_{T_1}^K} = \frac{g_1}{1 - \sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j} + \frac{\partial(C_Z^X - G_Z)}{\partial \Pi_{T_1}^K}$$

The sign of this derivative is ambiguous as well. The value of this derivative depends on the distribution of post-fiscal income and how the progressivity is changed (that is, the latter term in the derivative cannot be simplified any further in the general case).

Finally, the derivative with respect to the Kakwani index when this index is calculated with respect to the end income ranking of households can be calculated using equation 13 and transformation of the R-S index to Kakwani index, that is,

$$M_{T_1} = \left\{(G_X - C_X^Z) + \left(\frac{\sum_{i=1}^{n} g_i \Pi_{T_i}^{K^Z} + \sum_{j=1}^{m} b_j \rho_{B_j}^{K^Z}}{1 - \sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j}\right)\right\} - \left\{(G_X - C_X^{Z\backslash T_1}) + \left(\frac{\sum_{i=2}^{n} g_i \Pi_{T_i}^{K^{Z\backslash T_1}} + \sum_{j=1}^{m} b_j \rho_{B_j}^{K^{Z\backslash T_1}}}{1 - \sum_{i=2}^{n} g_i + \sum_{j=1}^{m} b_j}\right)\right\}$$

Therefore,

$$\frac{\partial M_{T_1}}{\partial \Pi_{T_1}^{K^Z}} = \frac{\partial(-C_X^Z)}{\Pi_{T_1}^{K^Z}} + \frac{g_1 + \sum_{i=2}^{n} g_i \frac{\partial \Pi_{T_i}^{K^Z}}{\partial \Pi_{T_1}^{K^Z}} + \sum_{j=1}^{m} b_j \frac{\partial \rho_{B_j}^{K^Z}}{\partial \Pi_{T_1}^{K^Z}}}{1 - \sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j}$$

Using the simplifying assumption that increase in the progressivity is unchanged in the final ranking (Z), the preceding derivative would be simplified to

$$\frac{\partial M_{T_1}}{\partial \Pi_{T_1}^{K^Z}} = \frac{g_1}{1 - \sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j}.$$

This derivative is always positive. Therefore, making a tax more progressive, when progressivity is calculated with respect to the end income ranking of households, is always equalizing (with or without reranking), assuming that no change in the end income ranking of households occurs as a result of a marginal increase in the progressivity of that tax. It is worth noting that the value of this derivative is equal to the one calculated previously[13] for the derivative of the marginal effect with respect to the traditional Kakwani index. This outcome is of course expected as these two types of Kakwani indexes are the same when there is no reranking.

6.2. Case of a Marginal Change in a Transfer

This section provides the derivatives of the marginal contribution of a transfer with respect to its relative size (b), the Kakwani index calculated with respect to the original income ranking of households ($\boldsymbol{\rho_B^K}$), and the Kakwani index calculated with respect to the end income ranking of households ($\boldsymbol{\rho_B^{K^Z}}$). Because there is no specific methodological difference between this section and the previous one, only the formulas for these derivatives are presented.

$$\frac{\partial M_{B_1}}{\partial b_1} = \frac{\partial(-C_X^Z)}{\partial b_1} + \frac{\left(\rho_{B_1}^{K^Z} + \frac{\partial \rho_{B_1}^{K^Z}}{\partial b_1} b_1 + \sum_{i=1}^{n} g_i \frac{\partial \Pi_{T_i}^{K^Z}}{\partial b_1} + \sum_{j=2}^{m} b_j \frac{\partial \rho_{B_j}^{K^Z}}{\partial b_1}\right) - (C_X^Z - G_Z)}{1 - \sum_{i=1}^{n} g_i + \sum_{j=1}^{m} b_j}$$

[13] Enami and others (2017).

With the simplifying assumption that the end income ranking of households (Z) would not change as a result of a marginal change in the relative size of the transfer, we have

$$\frac{\partial M_{B_1}}{\partial b_1}=\frac{\rho_{B_1}^{K^Z}-(C_X^Z-G_Z)}{1-\sum_{i=1}^{n}g_i+\sum_{j=1}^{m}b_j}=\frac{G_Z-C_{B_1}^Z}{1-\sum_{i=1}^{n}g_i+\sum_{j=1}^{m}b_j}$$

The sign of this derivative is ambiguous, but it would be positive if, for example, the concentration curve of a benefit (with respect to the end income ranking of households) happened to be above the Gini curve of the end income. Also, a negative concentration coefficient of a benefit (with respect to the end income ranking of households) would result in a positive sign for the preceding derivative, which happens when the poorer a household is, the higher the dollar value of the transfer it receives.

Alternatively, and using the traditional Kakwani index, we would have

$$\frac{\partial M_{B_1}}{\partial b_1}=\frac{\left(\rho_{B_1}^{RS}-G_X-C_{Z\backslash B_1}^X\right)\left(1-\sum_{i=1}^{n}g_i+\sum_{j=2}^{m}b_j\right)}{\left(1-\sum_{i=1}^{n}g_i+\sum_{j=1}^{m}b_j\right)^2}+\frac{\partial(C_Z^X-G_Z)}{\partial b_1}$$

The derivative with respect to ρ_B^K would be equal to

$$\frac{\partial M_{B_1}}{\partial \rho_B^K}=\frac{b_1}{1-\sum_{i=1}^{n}g_i+\sum_{j=1}^{m}b_j}+\frac{\partial(C_Z^X-G_Z)}{\partial \rho_B^K}$$

The sign of this derivative is ambiguous since the last term cannot be simplified any further.

Finally, the derivative with respect to $\rho_B^{K^Z}$ would be equal to

$$\frac{\partial M_{B_1}}{\partial \rho_B^{K^Z}}=\frac{\partial(-C_X^Z)}{\partial \rho_B^{K^Z}}+\frac{\left(b_1+\sum_{i=1}^{n}g_i\frac{\partial \Pi_{T_i}^{K^Z}}{\partial \rho_B^{K^Z}}+\sum_{j=2}^{m}b_j\frac{\partial \rho_{B_j}^{K^Z}}{\partial \rho_B^{K^Z}}\right)}{1-\sum_{i=1}^{n}g_i+\sum_{j=1}^{m}b_j}$$

Applying the simplifying assumption of no change in the final ranking (Z) results in the following formula:

$$\frac{\partial M_{B_1}}{\partial \rho_B^{K^Z}}=\frac{b_1}{1-\sum_{i=1}^{n}g_i+\sum_{j=1}^{m}b_j}$$

Unlike all preceding derivatives, this one has a positive sign, which means that making a transfer more progressive, when progressivity is calculated with respect to the end income ranking of households, will always reduce inequality as long as the end income ranking does not change. Similarly to the case of a tax explained in section 6.1, this derivative is equal to the one calculated in the previous paper for the derivative of the marginal contribution with respect to the Kakwani index in the absence of reranking in the system.

The main message of this paper is that in the presence of reranking, indicators of progressivity do not provide any insight into whether a tax or transfer reduces inequality in the marginal contribution sense. Mathematical derivations and various examples throughout this paper intended to make this message clear. The complicated and usually inconclusive math can be entirely avoided if the marginal contribution analysis is employed. In other words, there is no shortcut to answering fiscal policy questions other than performing simulations and accounting for all components (taxes and transfers) of a fiscal system.

7. Lambert Conundrum Revisited

Our previous work[14] introduced the Lambert conundrum in which a regressive tax exerts an equalizing effect. Similarly, a progressive tax can increase inequality. This paper shows that reranking can also result in a similar outcome specially for progressive taxes and transfers. Since reranking always happens in the real world, it is important to decompose the role of reranking in producing these odd outcomes from what one would describe as a pure Lambert conundrum. This section introduces a decomposition designed to achieve this goal.

To better introduce this decomposition technique, assume we are dealing with a regressive tax that has an equalizing effect. We would like to know how much change (reduction) in Gini happens before individuals are reranked and how much it happens after they are reranked.

$$MC_T = G_{X+B} - G_{X-T+B} = \overbrace{(G_{X+B} - G_{X-T^{NR}+B})}^{Change\ in\ Gini\ before\ reranking\ begins} + \overbrace{(G_{X-T^{NR}+B} - G_{X-T+B})}^{Change\ in\ Gini\ after\ reranking\ begines}$$

Where MC_T is the marginal contribution of a tax (we assume the system has only one tax and one transfer), G_{X+B} is the Gini before tax and G_{X-T+B} is the Gini after the tax is added to the fiscal system. Finally, $G_{X-T^{NR}+B}$ is the Gini of a simulated distribution of income in which we begin adding taxes to people but only up to the point that they are not reranked. The following example shows how this simulation works.

Table 58. Using an actual tax, T, to simulate a hypothetical tax, $\boldsymbol{T^{NR}}$, that does not create reranking

Individual	X+B	T	X+B-T	T^{NR}	X+B-T^{NR}
1	11	0	11	0	11
2	12	0	12	0	12
3	13	2	11	1	12
4	14	4	10	2	12

[14] Enami and others (2017).

In the pure Lambert conundrum, for example, the latter term of the decomposition equation above would be zero because there would be no reranking. Moreover, if the simulated tax, T^{NR}, is still regressive an equalizing we can conclude that the Lambert conundrum does not depend on the reranking. However, the size of the total reduction in Gini may significantly depend on the reranking and the above decomposition would identify the relative importance of it.

Generalizing this decomposition to the case of any tax or transfer in a fiscal system with numerous other taxes and transfers, we would have the following equations:

$$M_{T_1} = \left(G_{Z\setminus T_1} - G_{Z_{T_1^{NR}}}\right) + \left(G_{Z_{T_1^{NR}}} - G_Z\right)$$

$$M_{B_1} = \left(G_{Z\setminus B_1} - G_{Z_{B_1^{NR}}}\right) + \left(G_{Z_{B_1^{NR}}} - G_Z\right)$$

where Z is the end income (market income minus all taxes plus all transfers), $Z\setminus T_1$ ($Z\setminus B_1$) is the end income without including T_1 (B_1). Finally, $Z_{T_1^{NR}}$ ($Z_{B_1^{NR}}$) is the end income when the simulated T_1^{NR} (B_1^{NR}) is used instead of the actual T_1 (B_1).

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